The Combinatorialization of Linear Recurrences

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Solving Linear Recurrences

To solve a *k*th order linear recurrence

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} \quad (a_k \neq 0),$$

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$$h(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k$$

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If *h(x)* has *k* distinct roots $r_1, r_2, \ldots, r_k$,

then there exist constants $c_1, c_2, \ldots, c_k$ such that

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where $c_1, c_2, \ldots, c_k$ depend on the initial conditions.

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Goal: Prove this combinatorially

Example: The Fibonacci Recurrence

$$F_n = F_{n-1} + F_{n-2}$$

$$x^2 - x - 1 \text{ has roots}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$

$$\text{Thus, } F_n = c_1(\tfrac{1+\sqrt{5}}{2})^n + c_2(\tfrac{1-\sqrt{5}}{2})^n$$

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$$\left(F_0 = 0, F_1 = 1 \rightarrow c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = \frac{-1}{\sqrt{5}}\right)$$

Second Order Linear Recurrences

Suppose $h_0, h_1, h_2, \ldots$ satisfies for $n \geq 2$,

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} \quad (a_2 \neq 0),$$

If $\quad h(x) = x^2 - a_1 x - a_2$

has distinct roots r_1 and r_2,

then there exist constants c_1 and c_2 such that

$$h_n = c_1 r_1^n + c_2 r_2^n.$$

Second Order Linear Recurrences

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What does h_n count?

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Traditional model: h_n counts tilings of length n using **squares** and **dominoes**, where squares have a weight of a_1 and dominoes have a weight of a_2.

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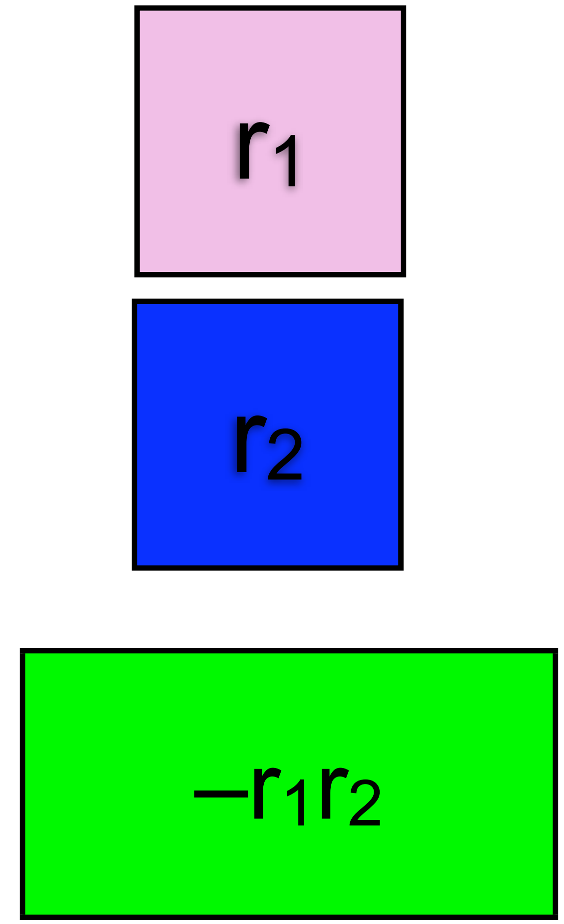
More colorful model:

We use two colors of squares:

Light squares have weight r_1

Dark squares have weight r_2

Dominoes have weight $-r_1 r_2$



The weight of a tiling is the *product* of the weights of its tiles.

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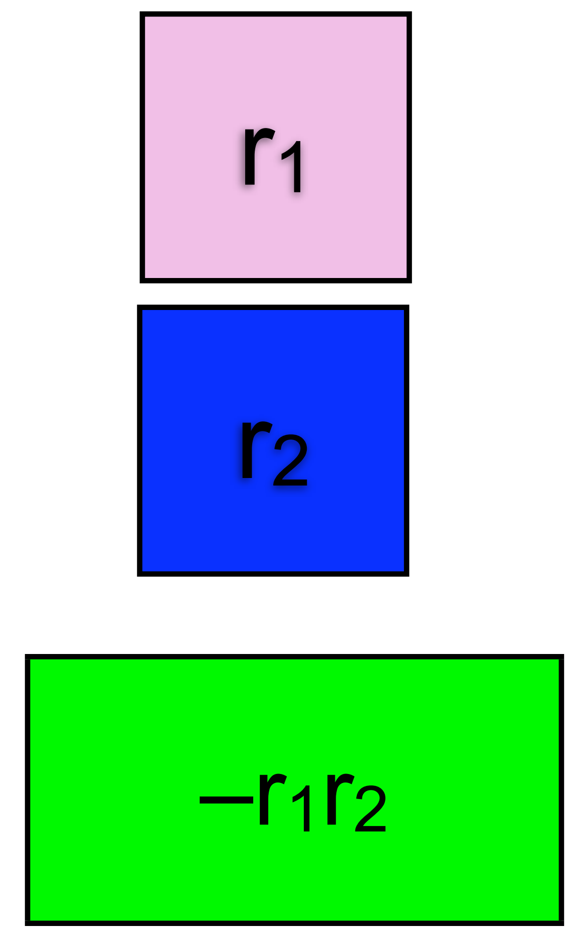

The weight of a tiling is the *product* of the weights of its tiles.

Initial tiles get a different weight than the other tiles.

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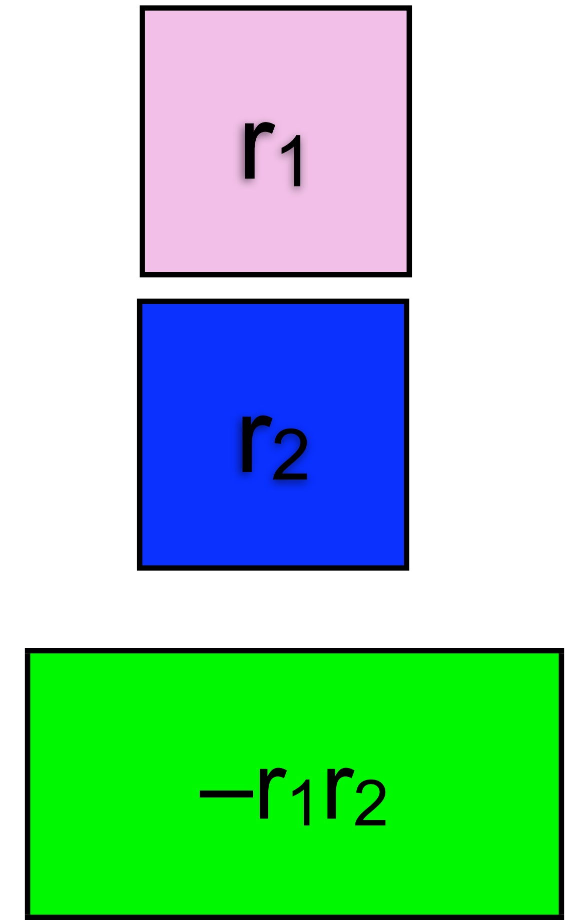


Initial Weights:

A square on cell 1 has weight c_1r_1 or c_2r_2.

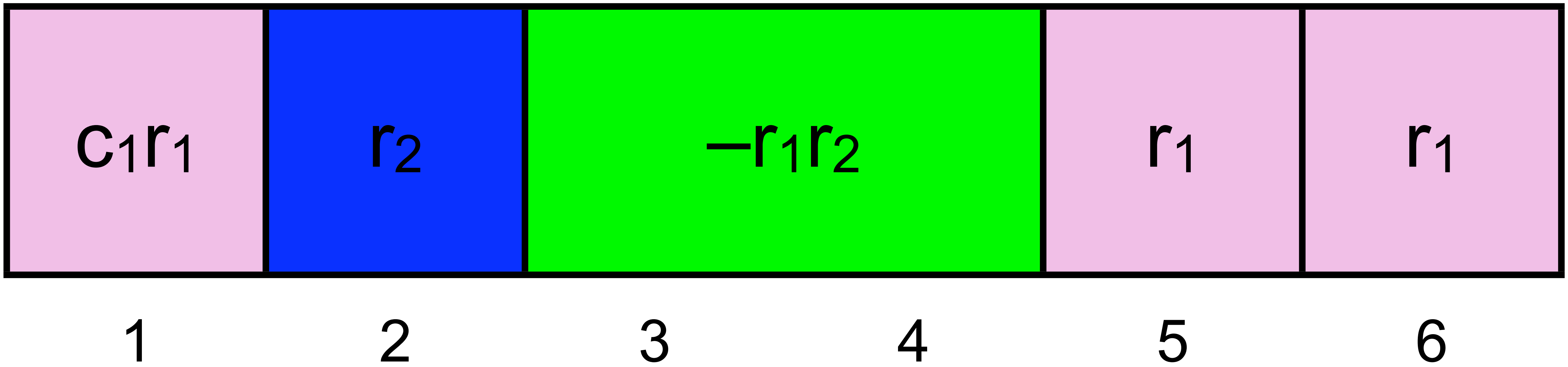
A domino on cell 1 has weight $-(c_1 + c_2)r_1r_2$.

The constants c_1 and c_2 will be determined later.

Example:

c_1r_1	r_2	$-r_1r_2$		r_1	r_1
1	2	3	4	5	6

Example:



has weight $-c_1r_1^4r_2^2$

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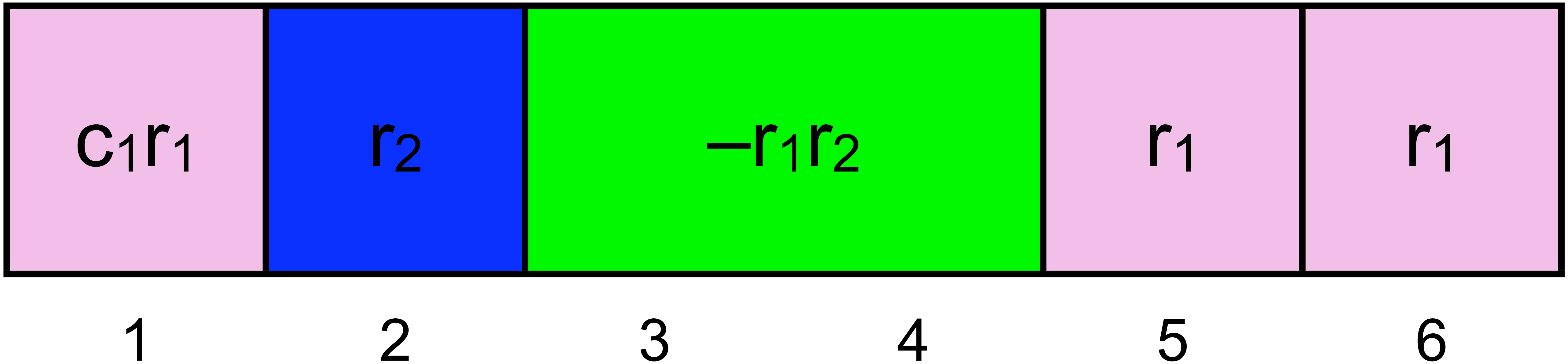

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Let W_n be the total weight of all tilings of length *n*.

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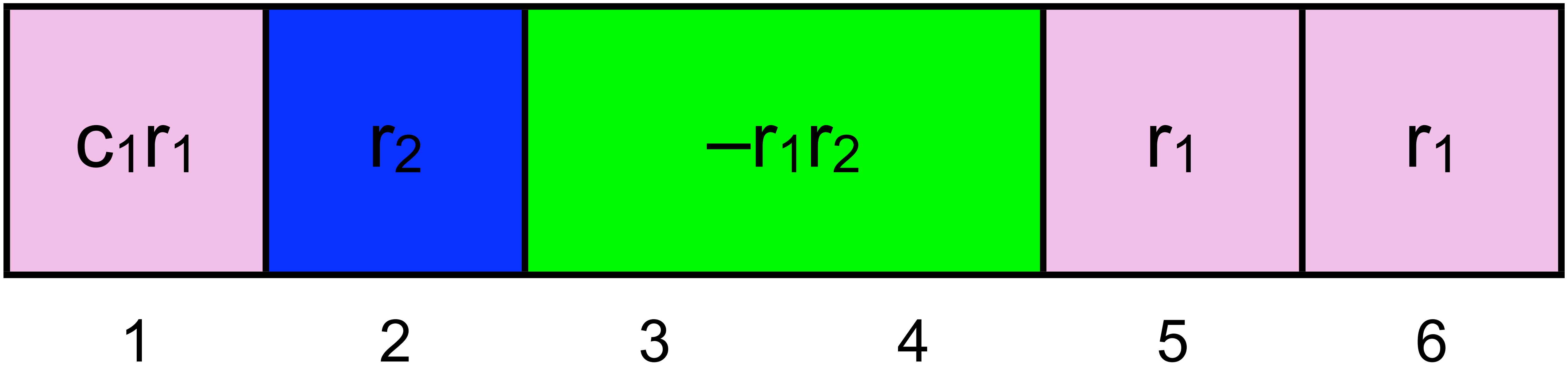

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Let W_n be the total weight of all tilings of length *n*.

Claim: W_n satisfies the same recurrence as h_n.

For $n > 2$,

$$W_n = r_1 W_{n-1} + r_2 W_{n-1} - r_1 r_2 W_{n-2}$$

Total weight

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$$W_n = r_1 W_{n-1} + r_2 W_{n-1} - r_1 r_2 W_{n-2}$$

Total weight Ends in light square

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Total weight Ends in light square Ends in dark square

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Total weight Ends in light square Ends in dark square Ends in domino

$$= (r_1 + r_2) W_{n-1} - r_1 r_2 W_{n-2}$$

$$= a_1 W_{n-1} + a_2 W_{n-2}$$

For $n > 2$,

$$W_n = r_1 W_{n-1} + r_2 W_{n-1} - r_1 r_2 W_{n-2}$$

Total weight Ends in light square Ends in dark square Ends in domino

$$= (r_1 + r_2)\, W_{n-1} - r_1 r_2 W_{n-2}$$

$$= a_1 W_{n-1} + a_2 W_{n-2}$$

since $x^2 - a_1 x - a_2 = (x - r_1)(x - r_2)$

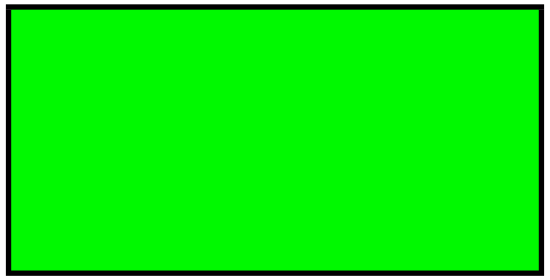
$$= x^2 - (r_1 + r_2)x + r_1 r_2$$

Why does $W_n = c_1 r_1^n + c_2 r_2^n$?

Combinatorial Proof:

Definition: A tiling is impure if it contains a domino or if it contains two adjacent squares of opposite color.

Thus a tiling has an impurity if it contains



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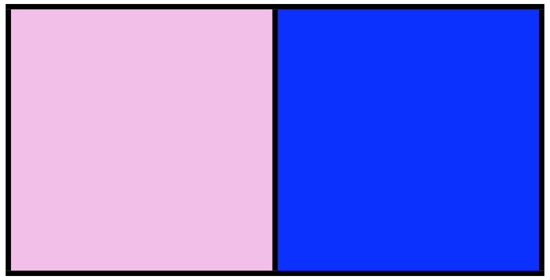

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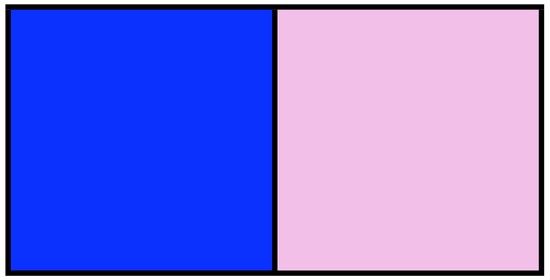
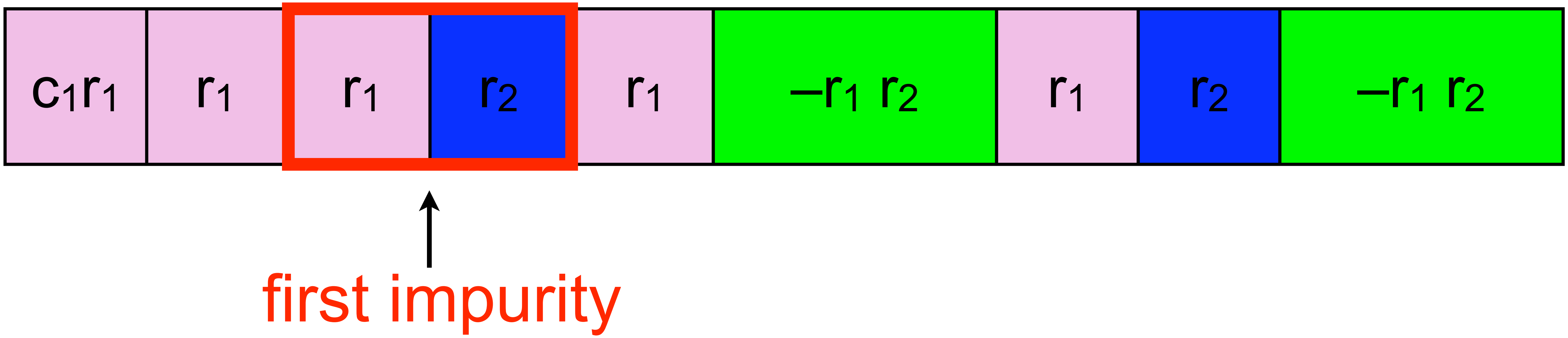


Claim: Impure Tilings Sum to Zero

If the tiling does not start with an impurity



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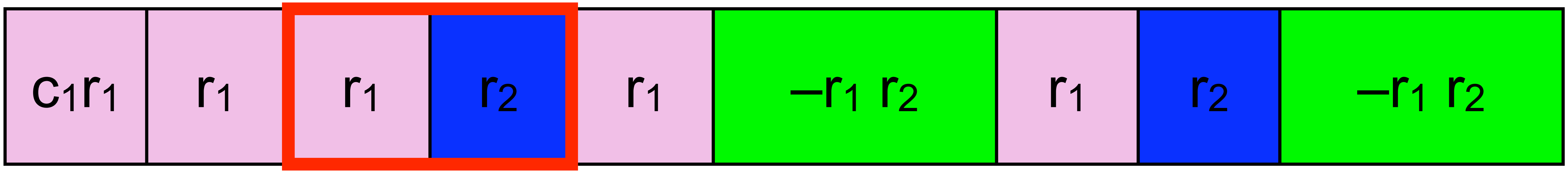

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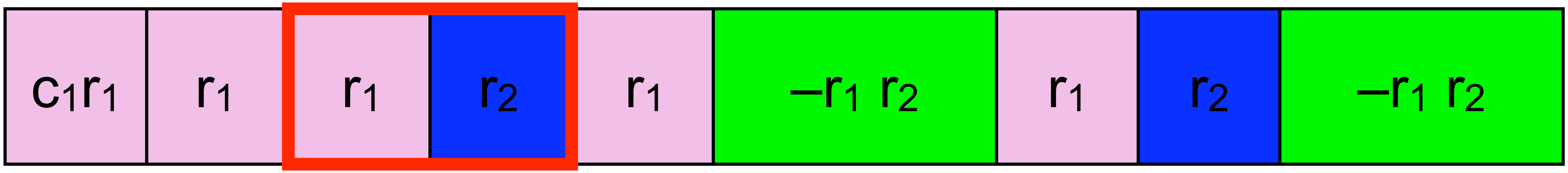

Find a mate of opposite weight!

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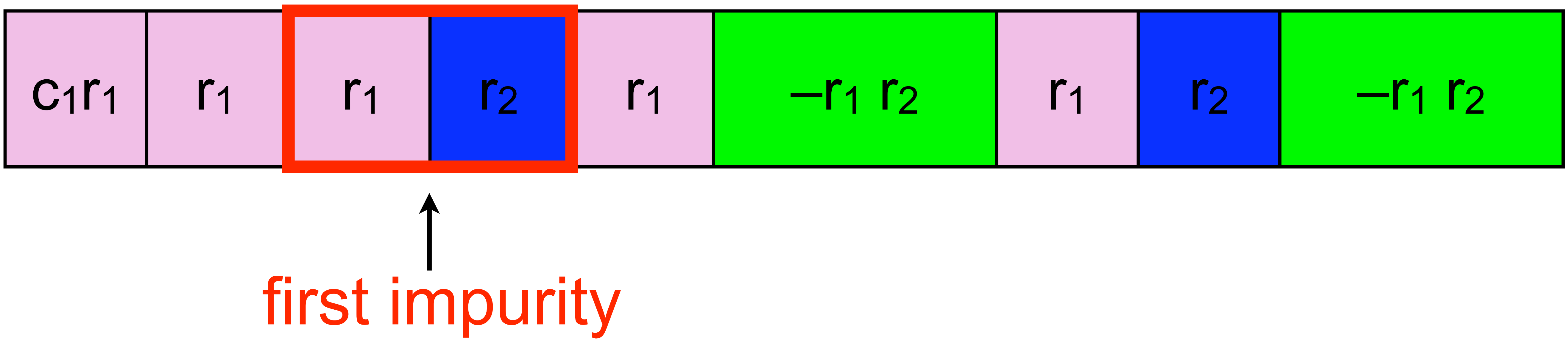

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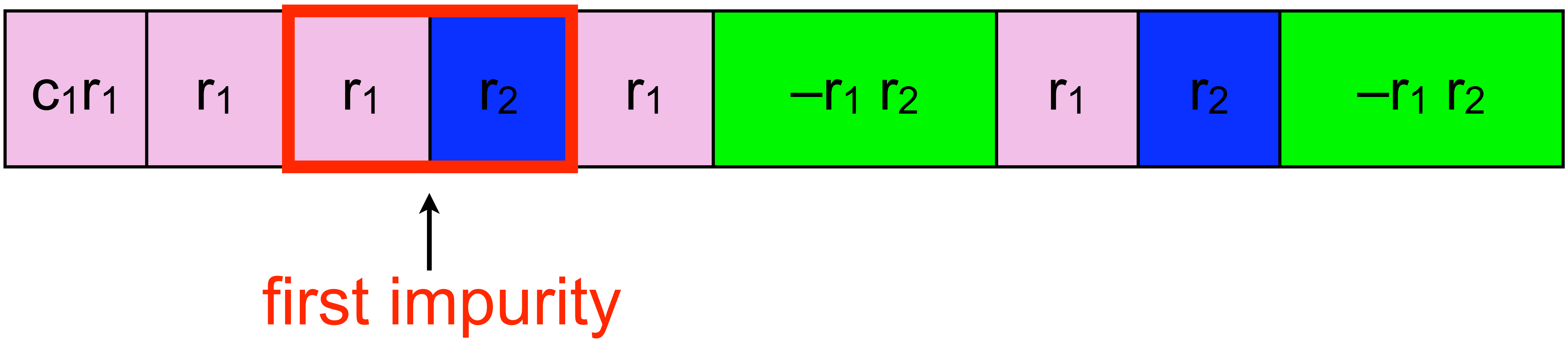

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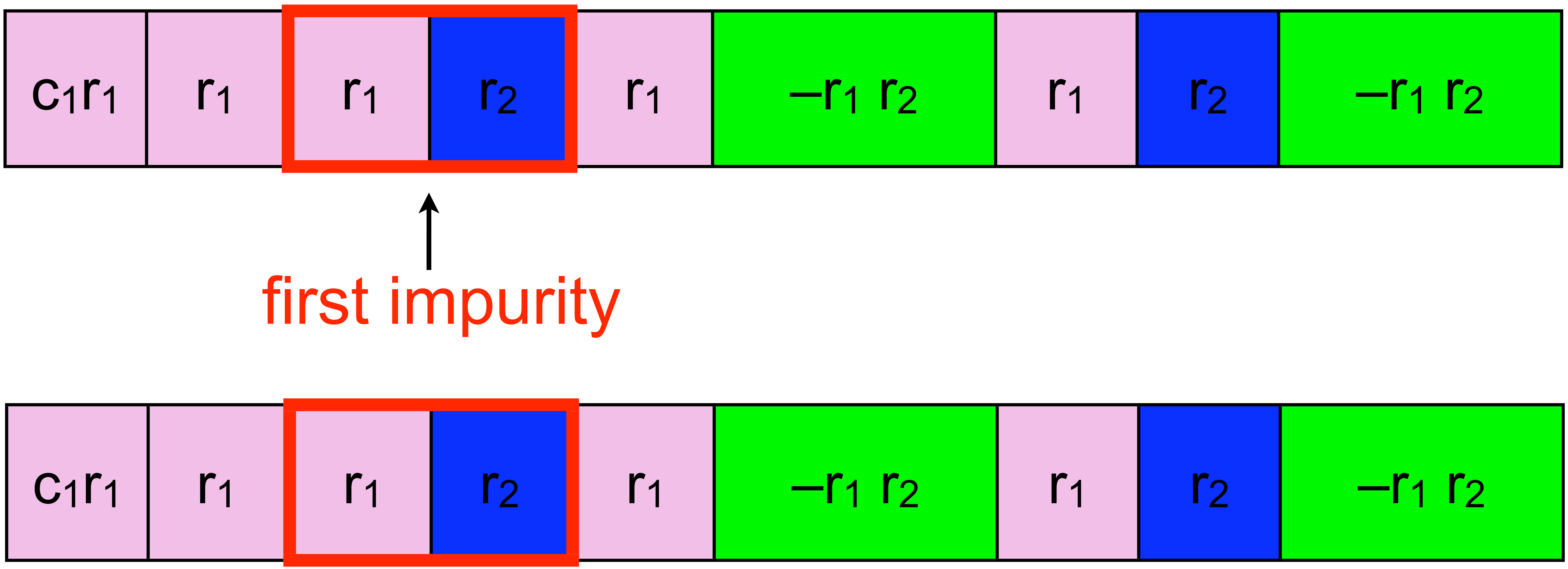

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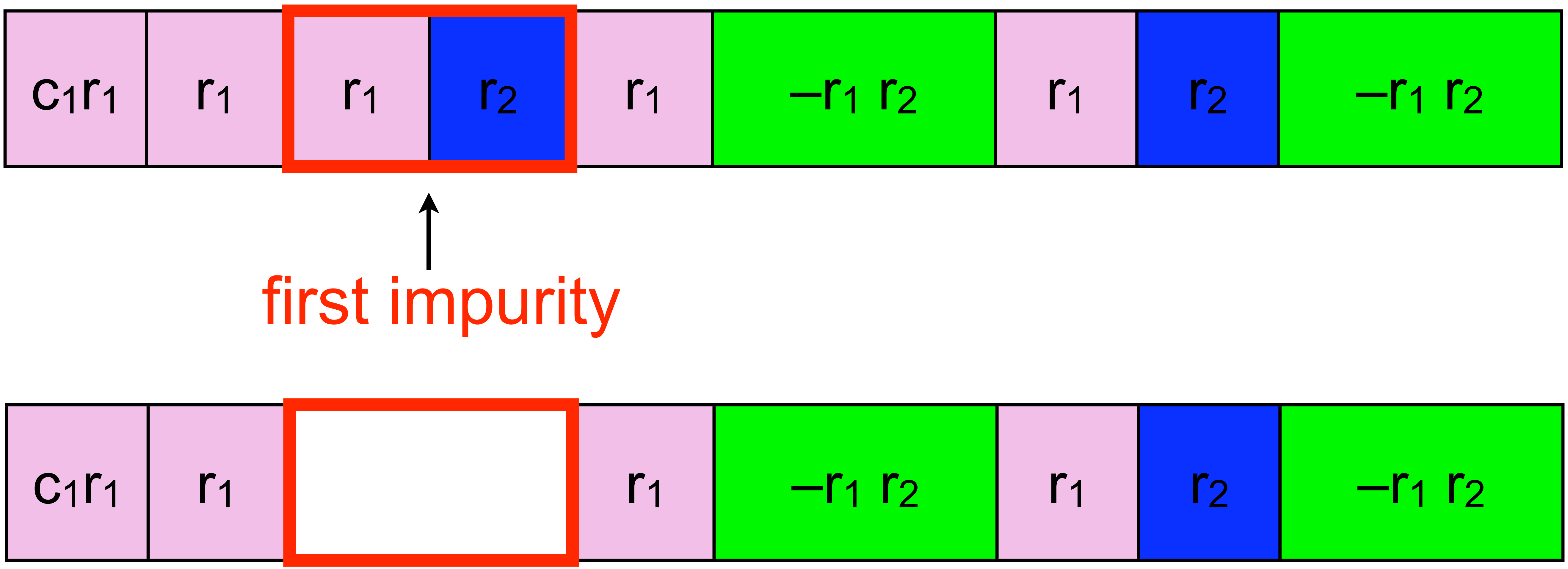

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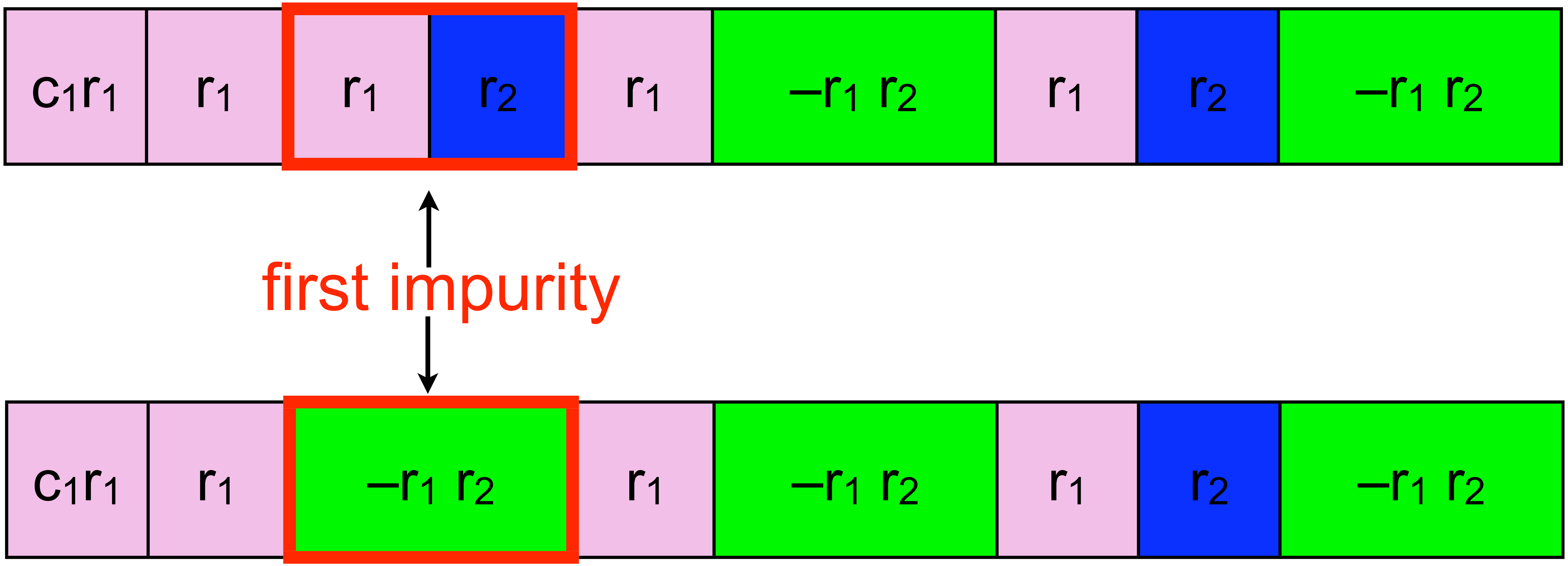

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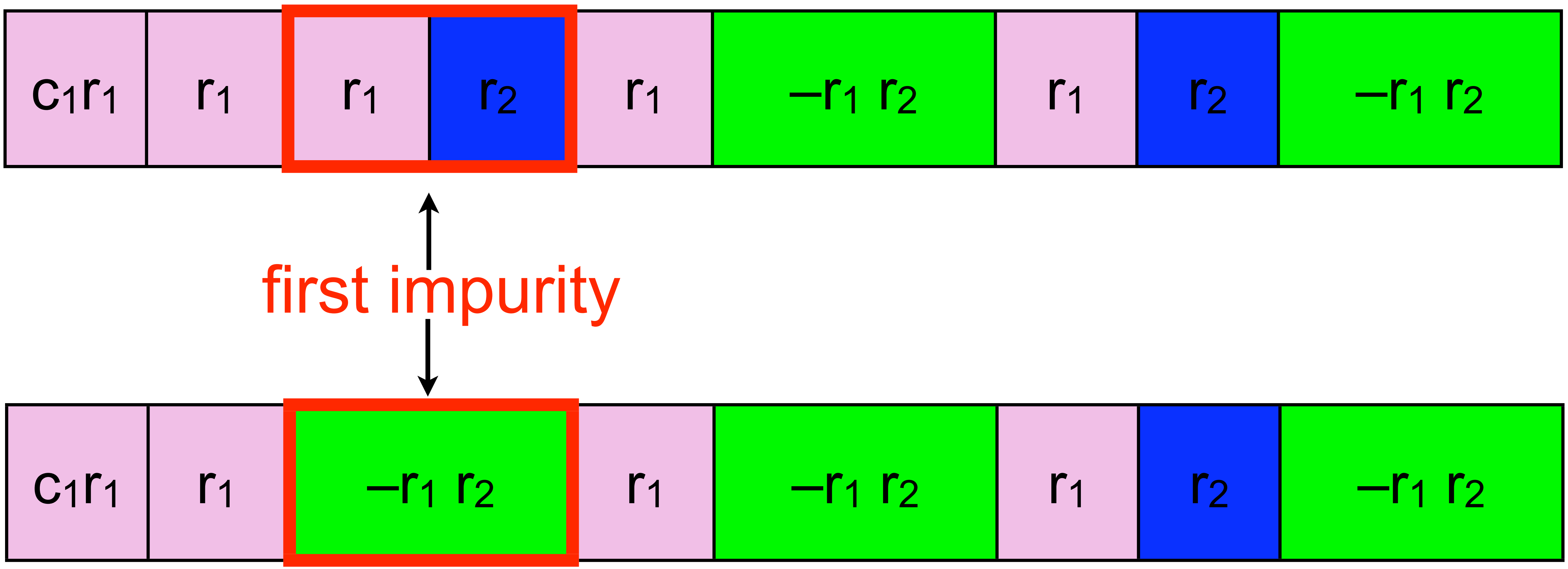
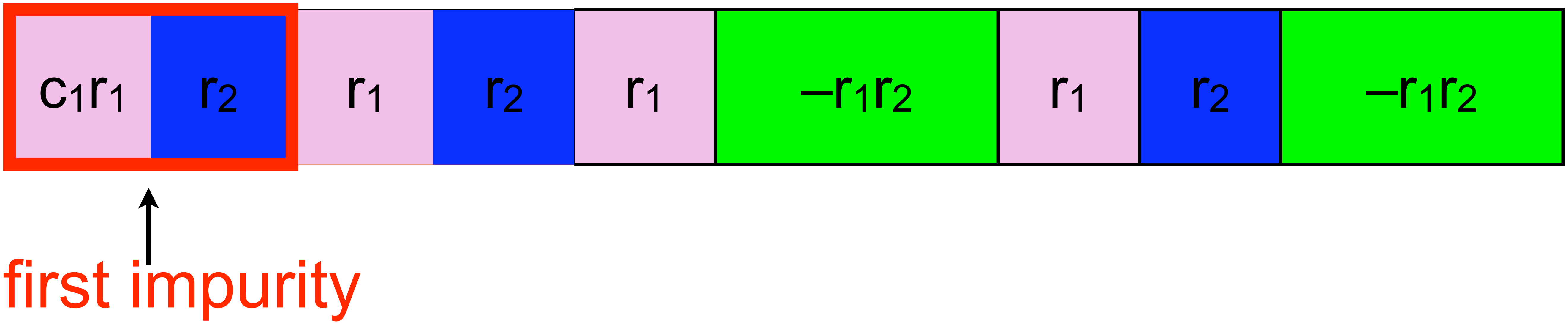


Weights sum to zero!

Impure Tilings Sum to Zero

If the tiling does start with an impurity



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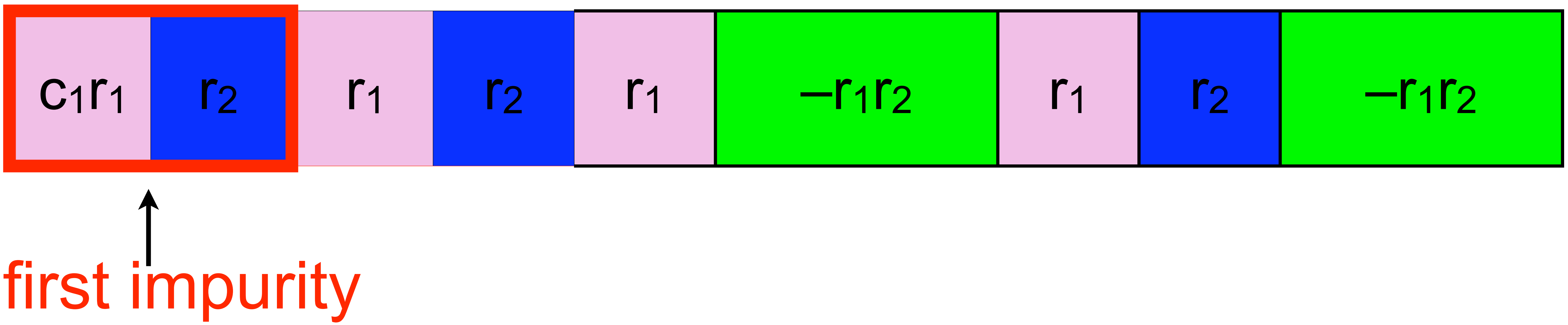
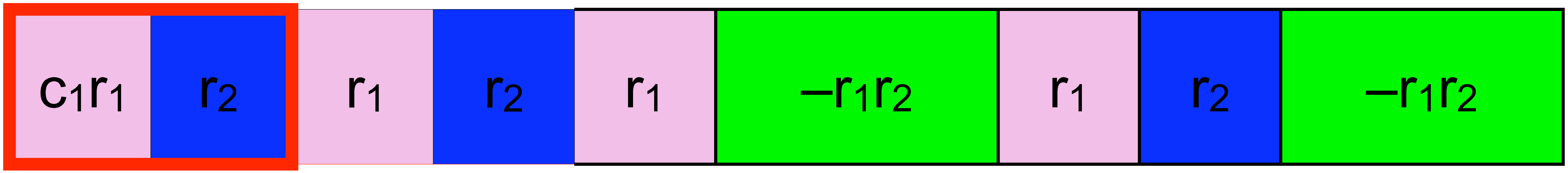


Find a trio that sums to zero!

Impure Tilings Sum to Zero

If the tiling does start with an impurity



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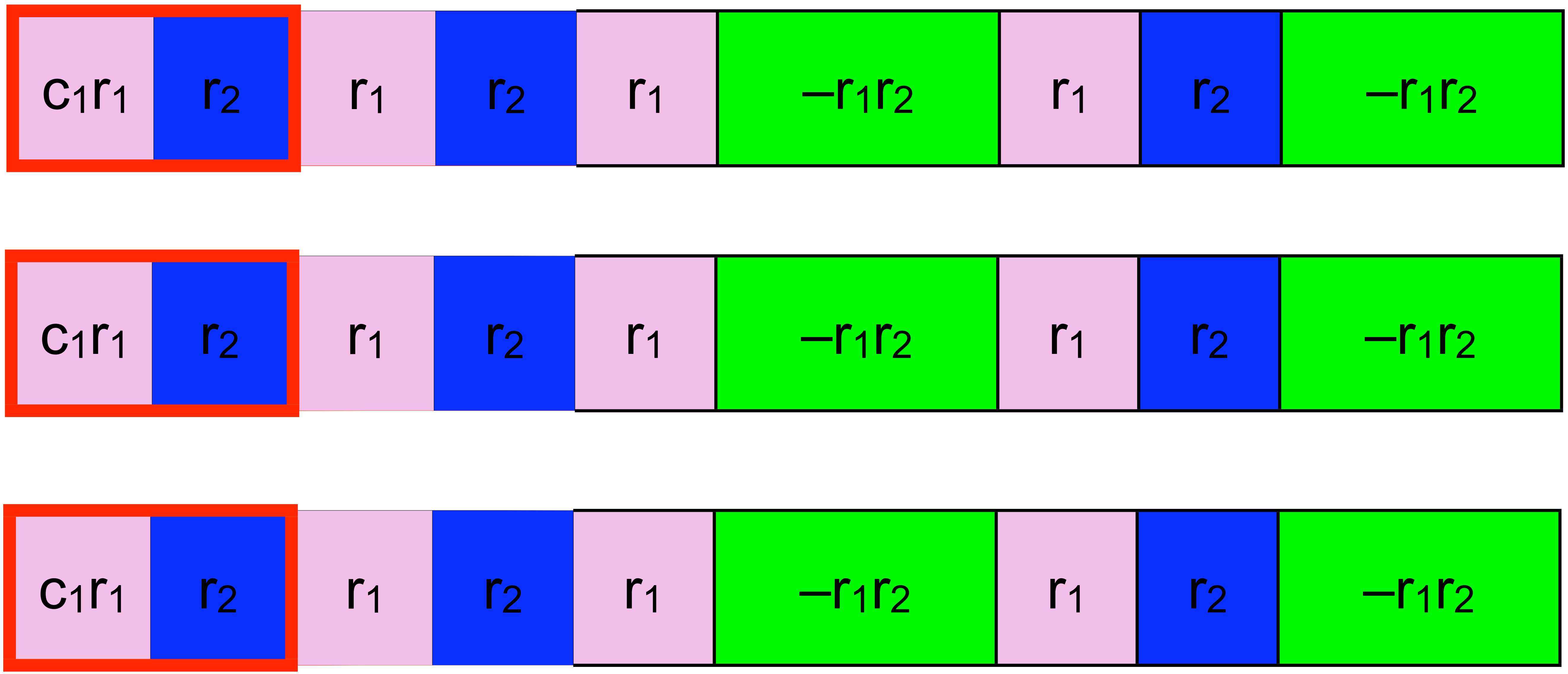

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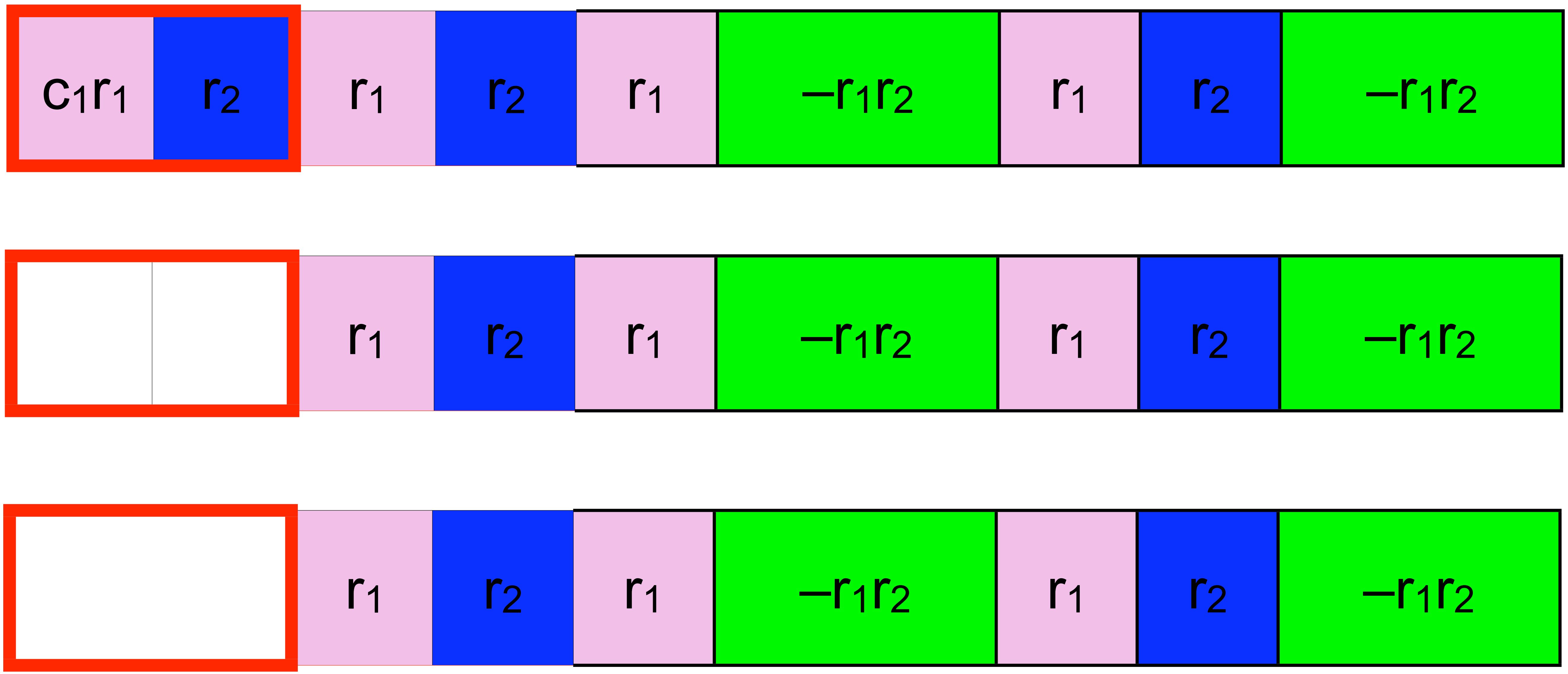

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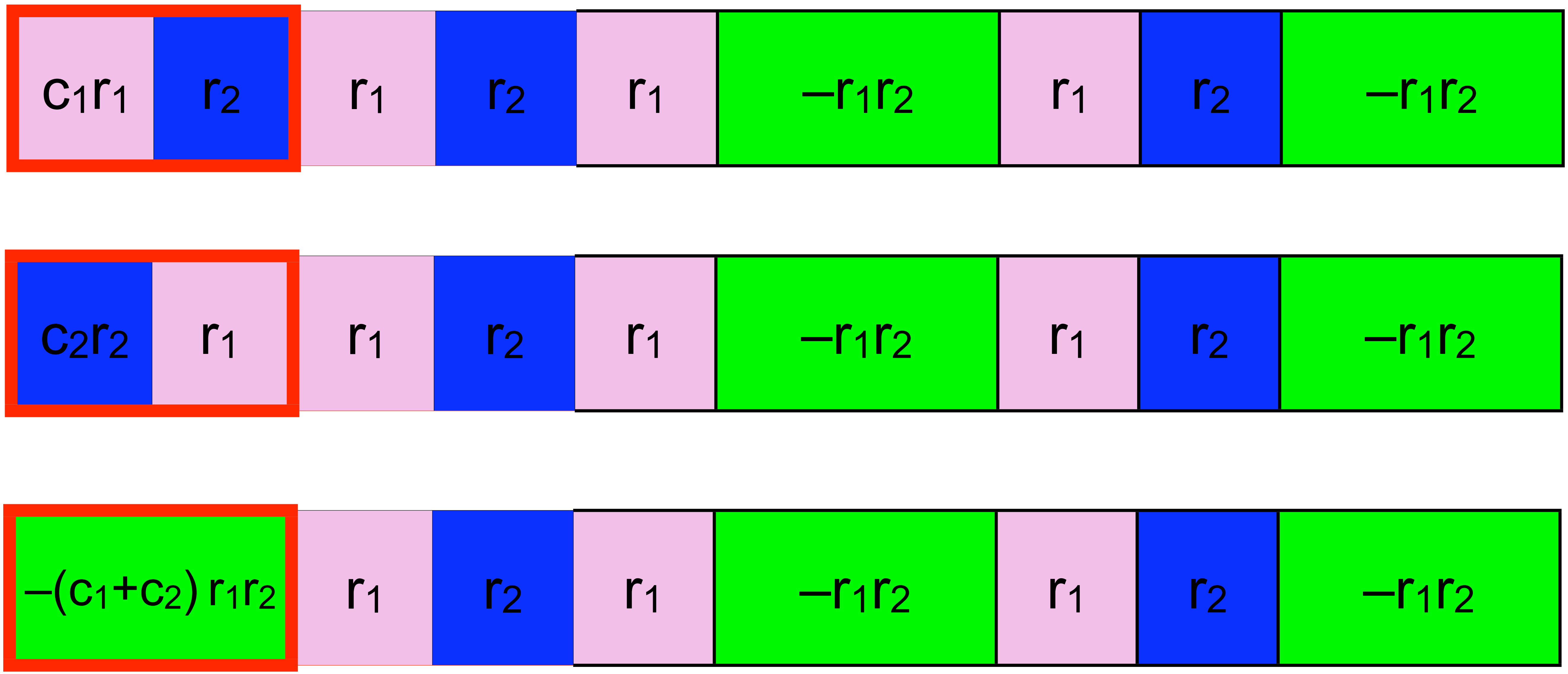

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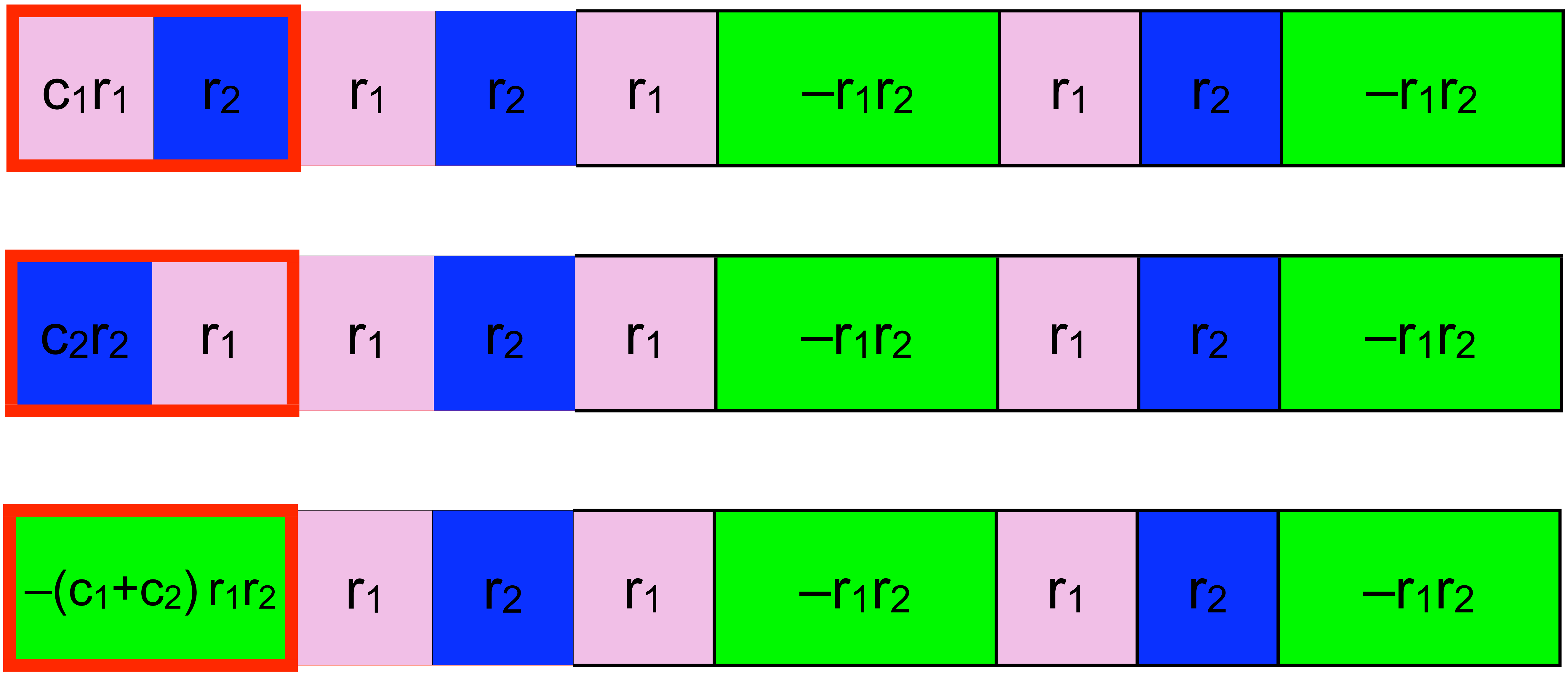

$$c_1r_1r_2 + c_2r_2r_1 - (c_1+c_2)\, r_1r_2 = 0$$

Since W_n is the total weight of all tilings, and since the impure tilings sum to zero, Since W_n is the total weight of all tilings, and since the impure tilings sum to zero,

W_n is the total weight of all pure tilings.

The pure tilings

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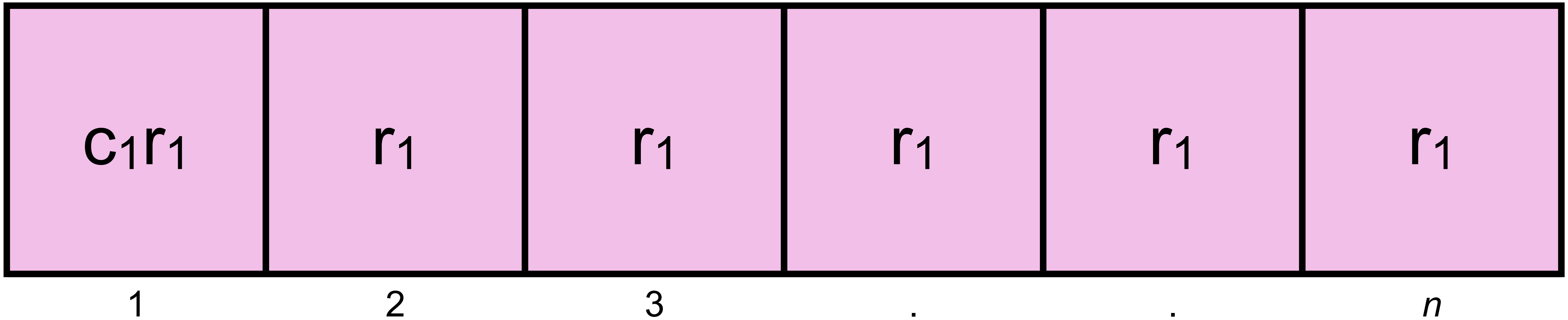
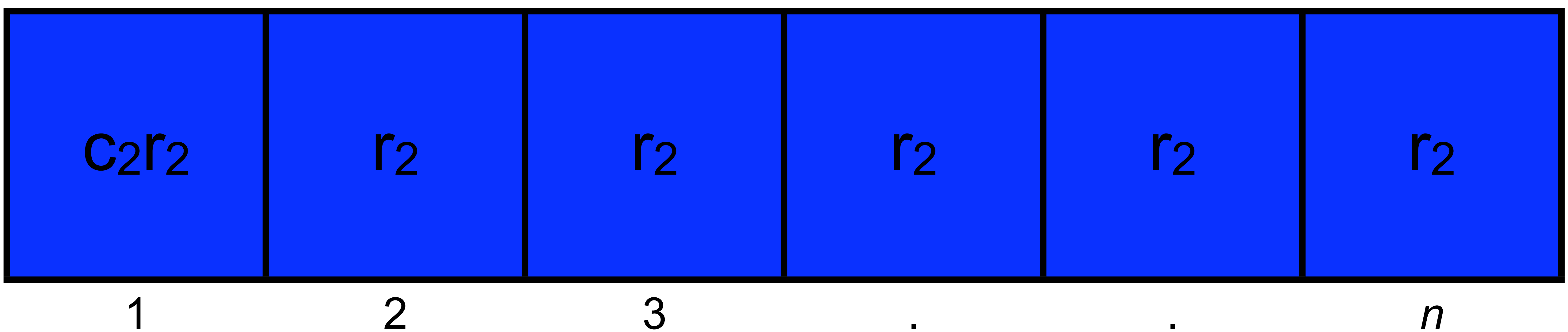
c_1r_1	r_1	r_1	r_1	r_1	r_1
1	2	3	.	.	*n*

The pure tilings





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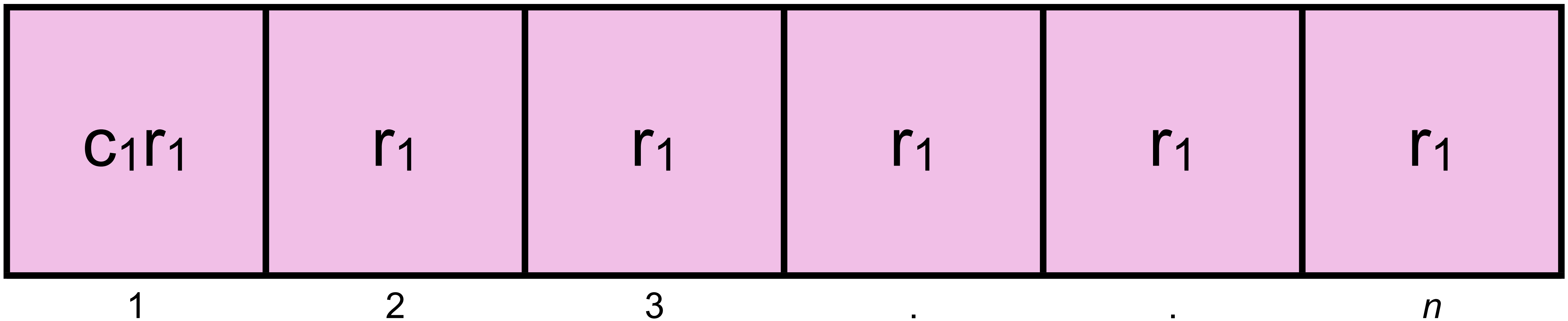
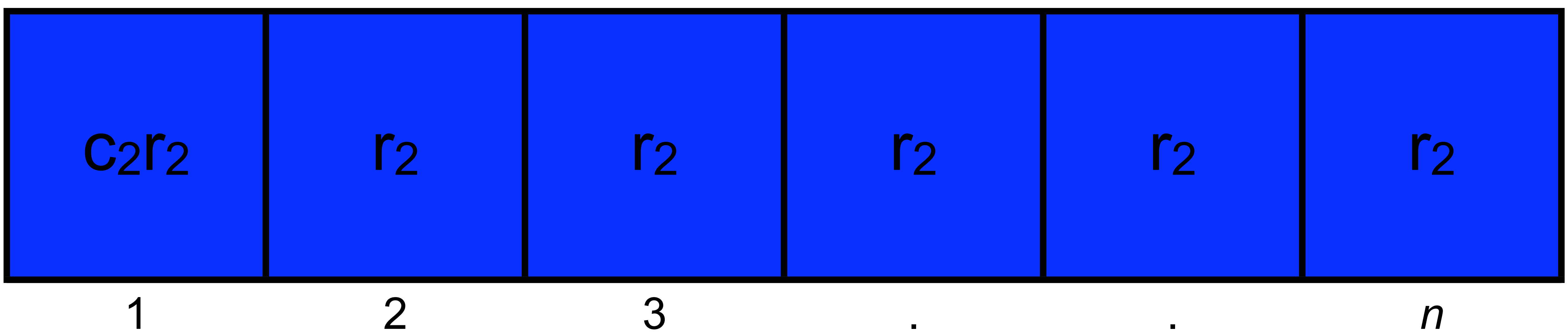


weight = $c_1 r_1^n$



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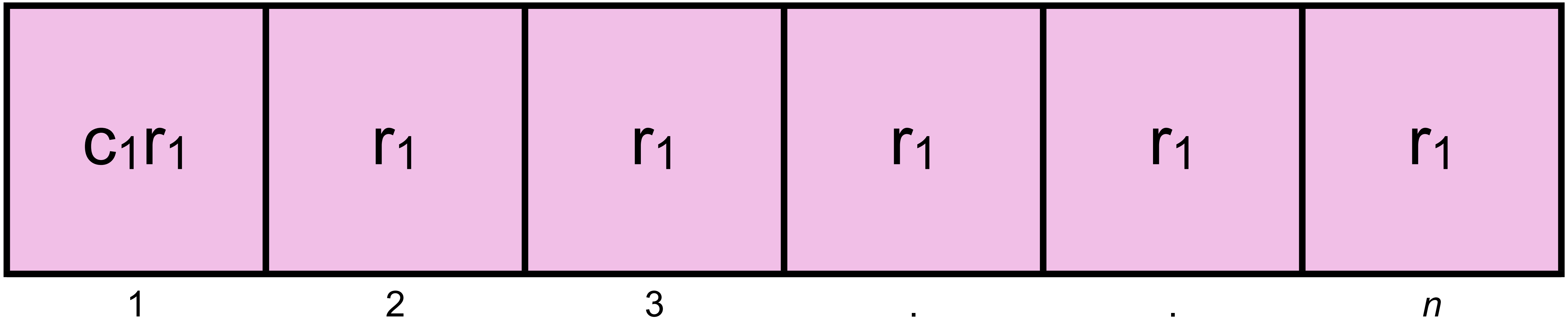

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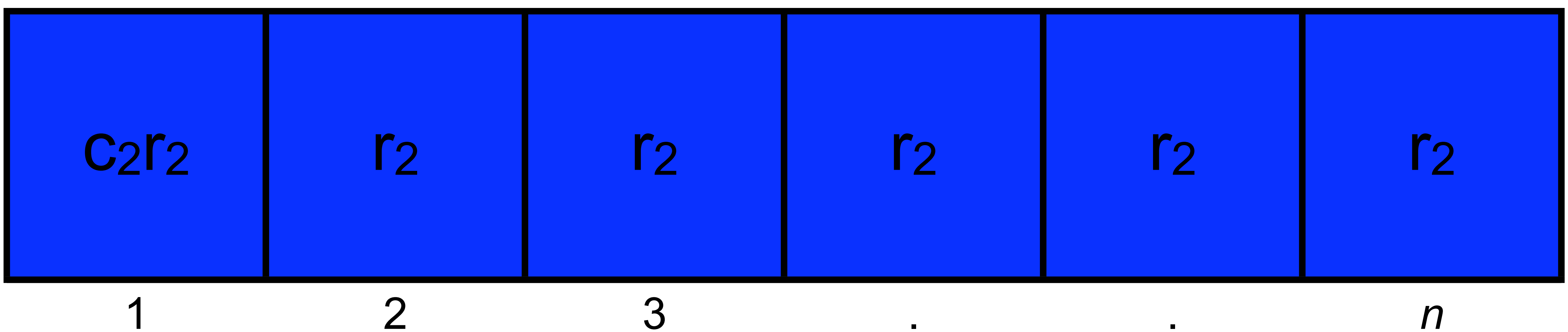
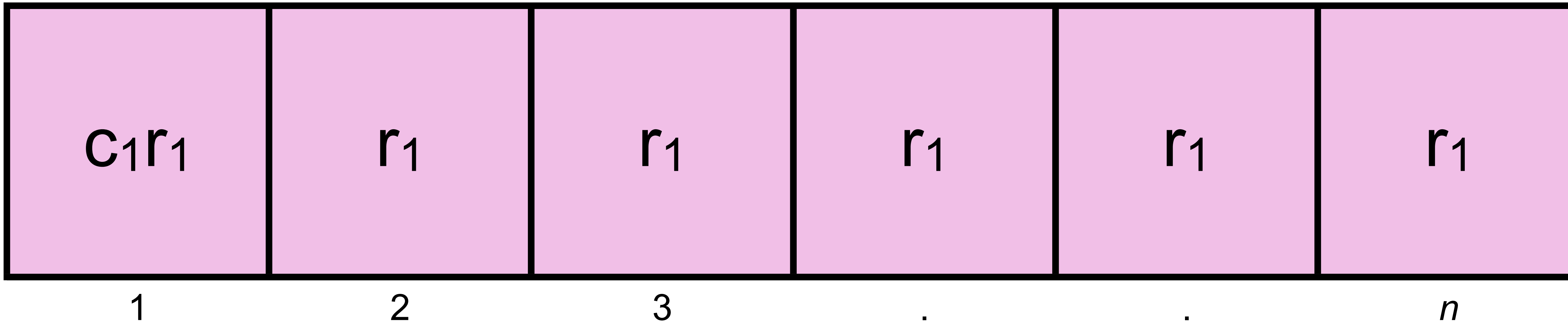
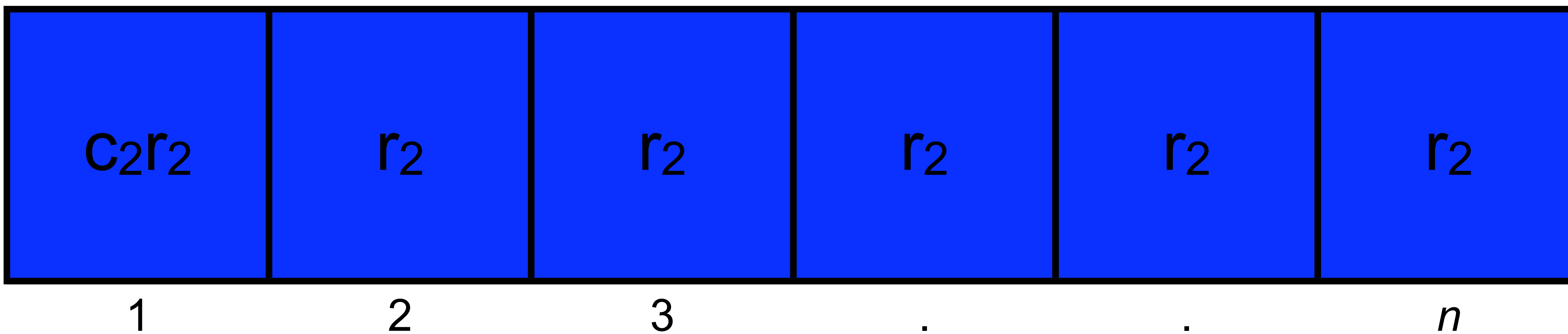


weight = $c_2r_2^n$

The pure tilings



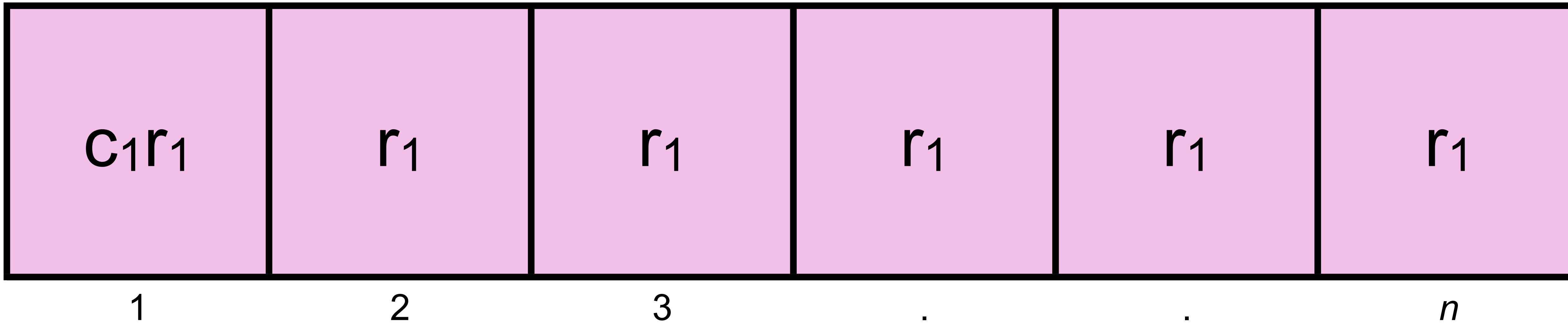
weight = $c_1 r_1^n$



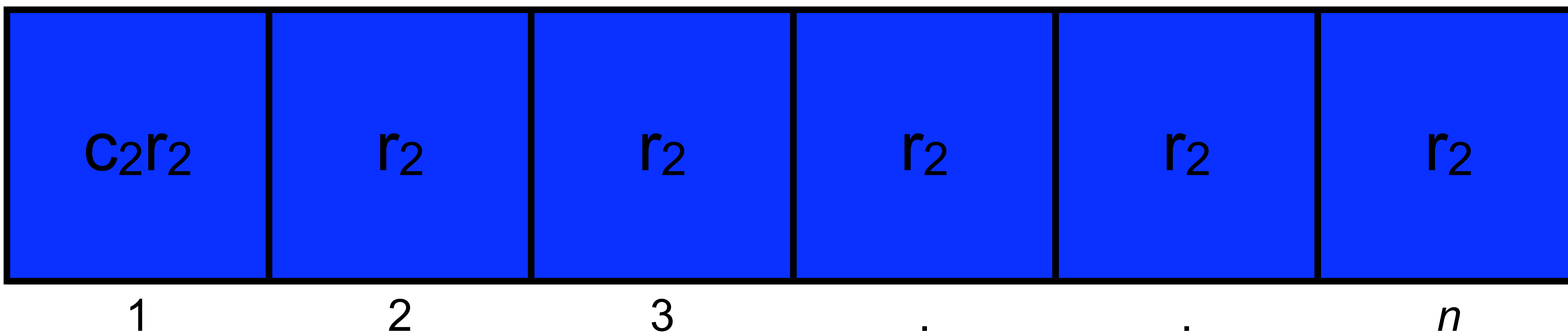
weight = $c_2 r_2^n$

W_n = Total Weight = $c_1 r_1^n + c_2 r_2^n$

The pure tilings



weight = $c_1r_1^n$



weight = $c_2r_2^n$

W_n = Total Weight = $c_1r_1^n + c_2r_2^n$

Since h_n and W_n satisfy the same recurrence, they will be equal if they have the same initial conditions. Thus, we choose c_1 and c_2 so that $W_0 = h_0$ and $W_1 = h_1$.

Thus, we solve

$$\begin{pmatrix} 1 & 1 \\ r_1 & r_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}$$

Since the matrix has determinant $r_2 - r_1 \neq 0$, there are unique constants c_1 and c_2 such that

$$h_n = W_n = c_1 r_1^n + c_2 r_2^n$$

Third Order Linear Recurrences

Suppose $h_0, h_1, h_2, \ldots$ satisfies for $n \geq 3$,

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + a_3 h_{n-3} \quad (a_3 \neq 0),$$

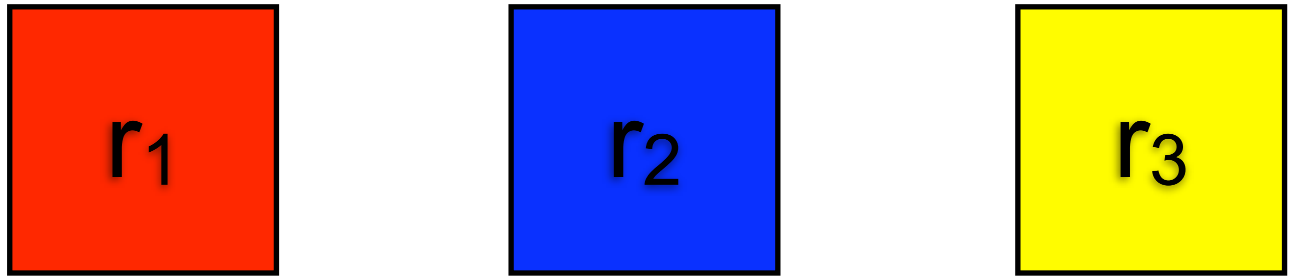
If $h(x) = x^3 - a_1 x^2 - a_2 x - a_3$

has distinct roots r_1, r_2, and r_3,

then there exist constants c_1, c_2, and c_3 such that

$$h_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n.$$

We use three types of squares:



We use three types of squares:

r_1 r_2 r_3

We use three types of dominoes:

$-r_1 r_2$ $-r_1 r_3$ $-r_2 r_3$

We use three types of squares:

r_1 r_2 r_3

We use three types of dominoes:

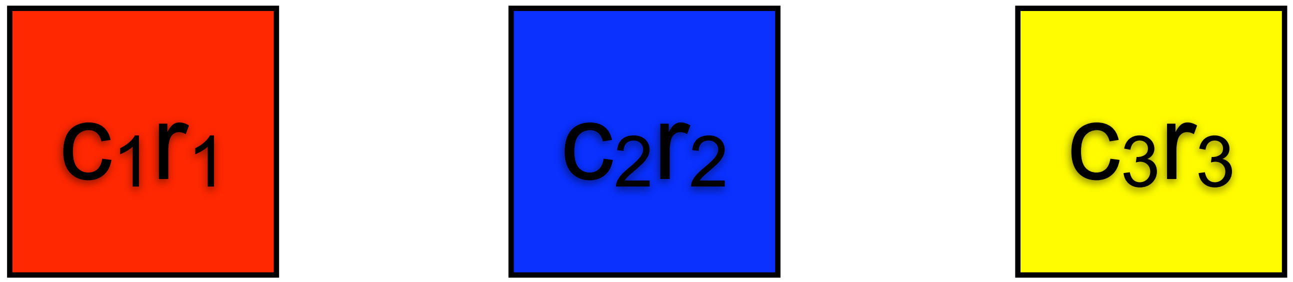
$-r_1r_2$ $-r_1r_3$ $-r_2r_3$

And one type of tromino:

$r_1r_2r_3$

Initial tiles get a different weight:

We use three types of squares:



Initial tiles get a different weight:

We use three types of squares:

$c_1 r_1$ $\qquad$ $c_2 r_2$ $\qquad$ $c_3 r_3$

We use three types of dominoes:

$-(c_1+c_2)r_1r_2$ $\qquad$ $-(c_1+c_3)\, r_1r_3$ $\qquad$ $-(c_2+c_3)\, r_2r_3$

Initial tiles get a different weight:

We use three types of squares:

$c_1 r_1$ $\quad$ $c_2 r_2$ $\quad$ $c_3 r_3$

We use three types of dominoes:

$-(c_1+c_2)r_1 r_2$ $\quad$ $-(c_1+c_3)\, r_1 r_3$ $\quad$ $-(c_2+c_3)\, r_2 r_3$

And one type of tromino:

$(c_1+c_2+c_3)r_1 r_2 r_3$

Let W_n be the total weight of all n-tilings.

W_n satisfies the same recurrence as h_n, since

$$W_n = (r_1 + r_2 + r_3)\,W_{n-1} - (r_1 r_2 + r_1 r_3 + r_2 r_3)W_{n-2} + r_1 r_2 r_3 W_{n-3}$$

Total weight Ends in a square Ends in a domino Ends in a tromino

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$$= a_1 W_{n-1} + a_2 W_{n-2} + a_3 W_{n-3}$$

$$\text{since } x^3 - a_1 x^2 - a_2 x - a_3 = (x - r_1)(x - r_2)(x - r_3)$$

$$= x^3 - (r_1 + r_2 + r_3) x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3) x - r_1 r_2 r_3$$

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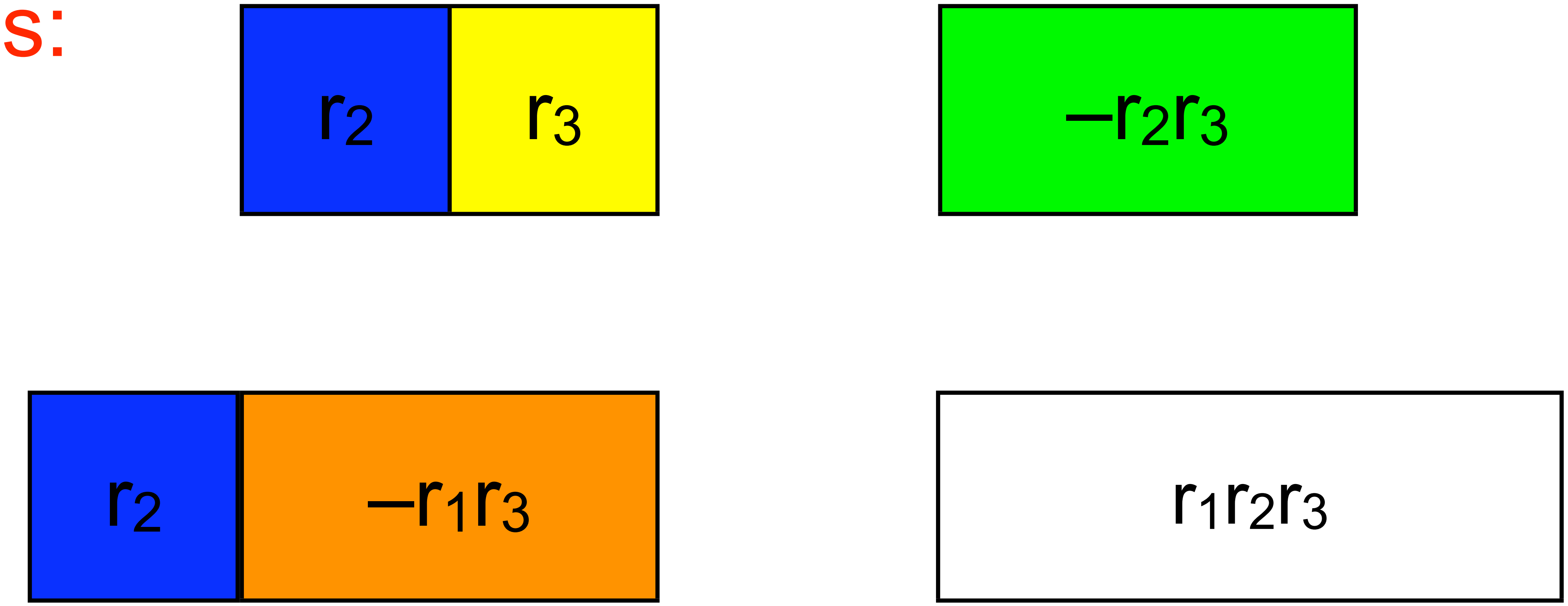

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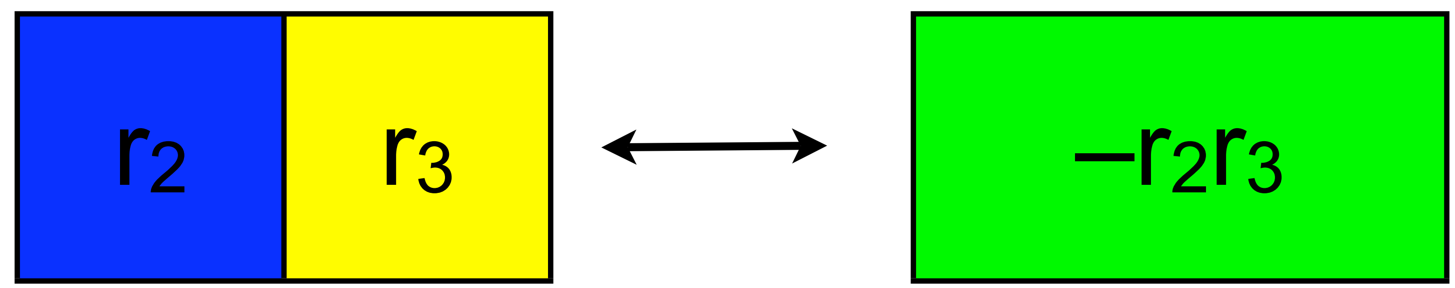

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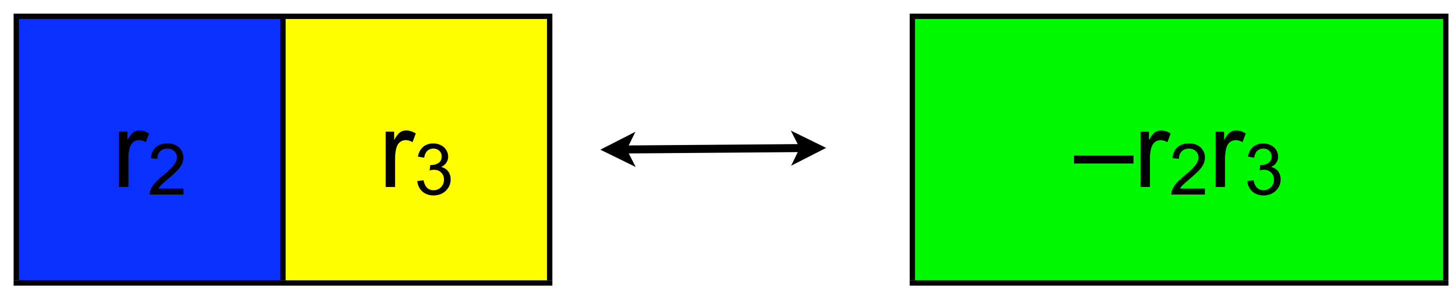

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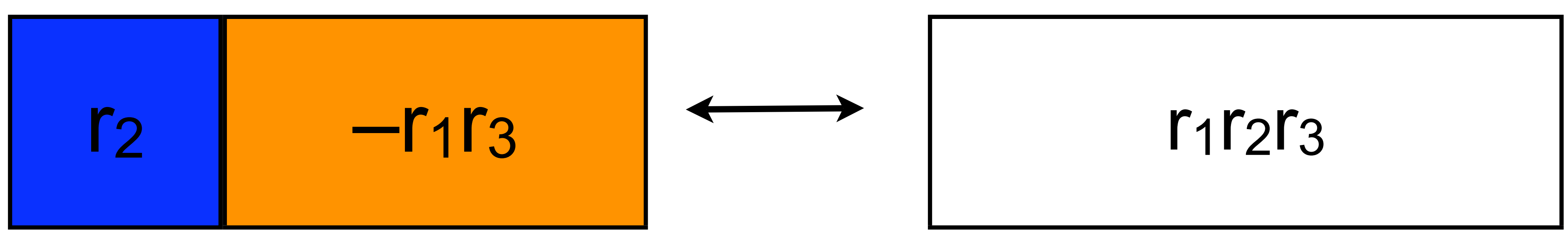

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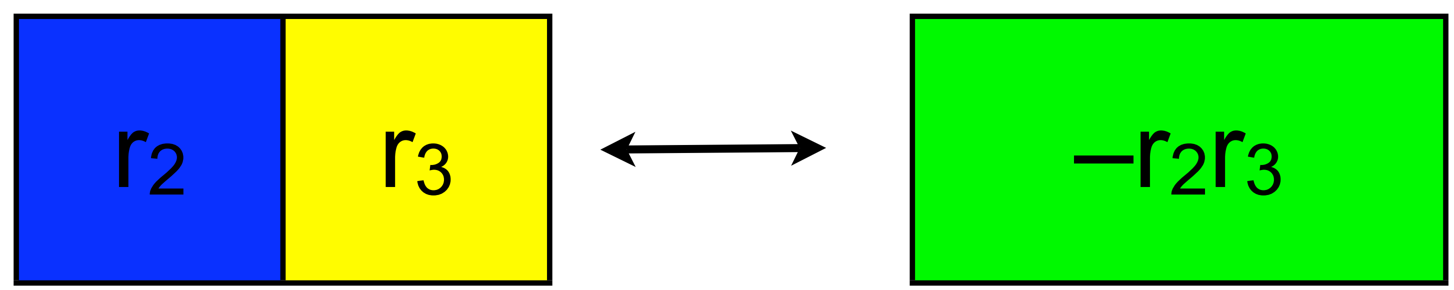

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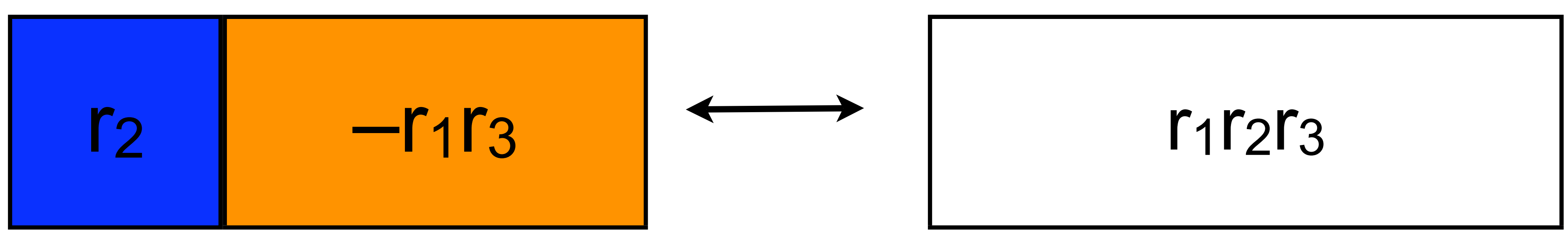

Non-initial impurities sum to zero!

Initial impurities sum to zero too!

$$-(c_1+c_2)r_1r_2 + c_1r_1 \; r_2 + c_2r_2 \; r_1 = 0$$

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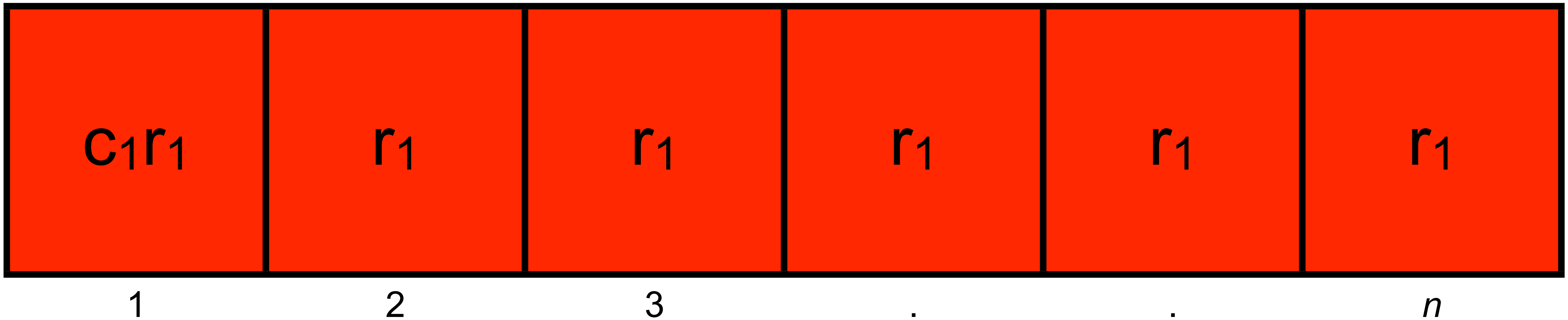

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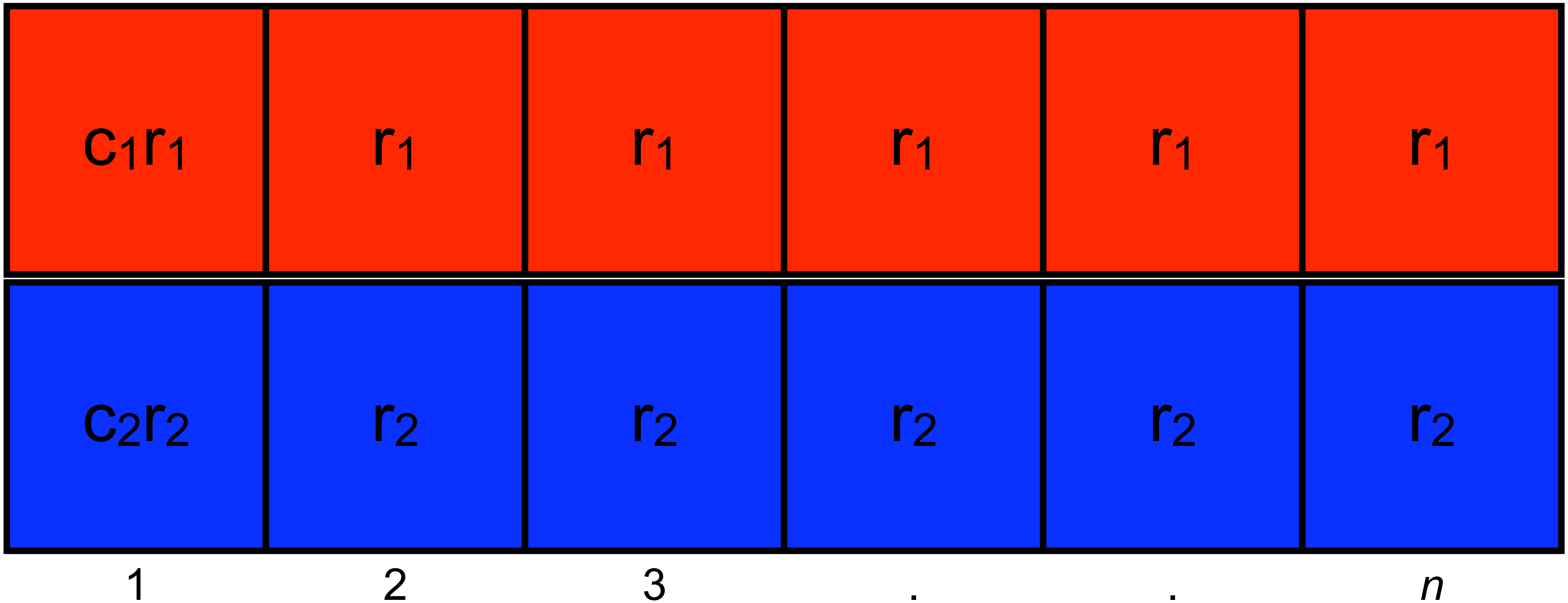

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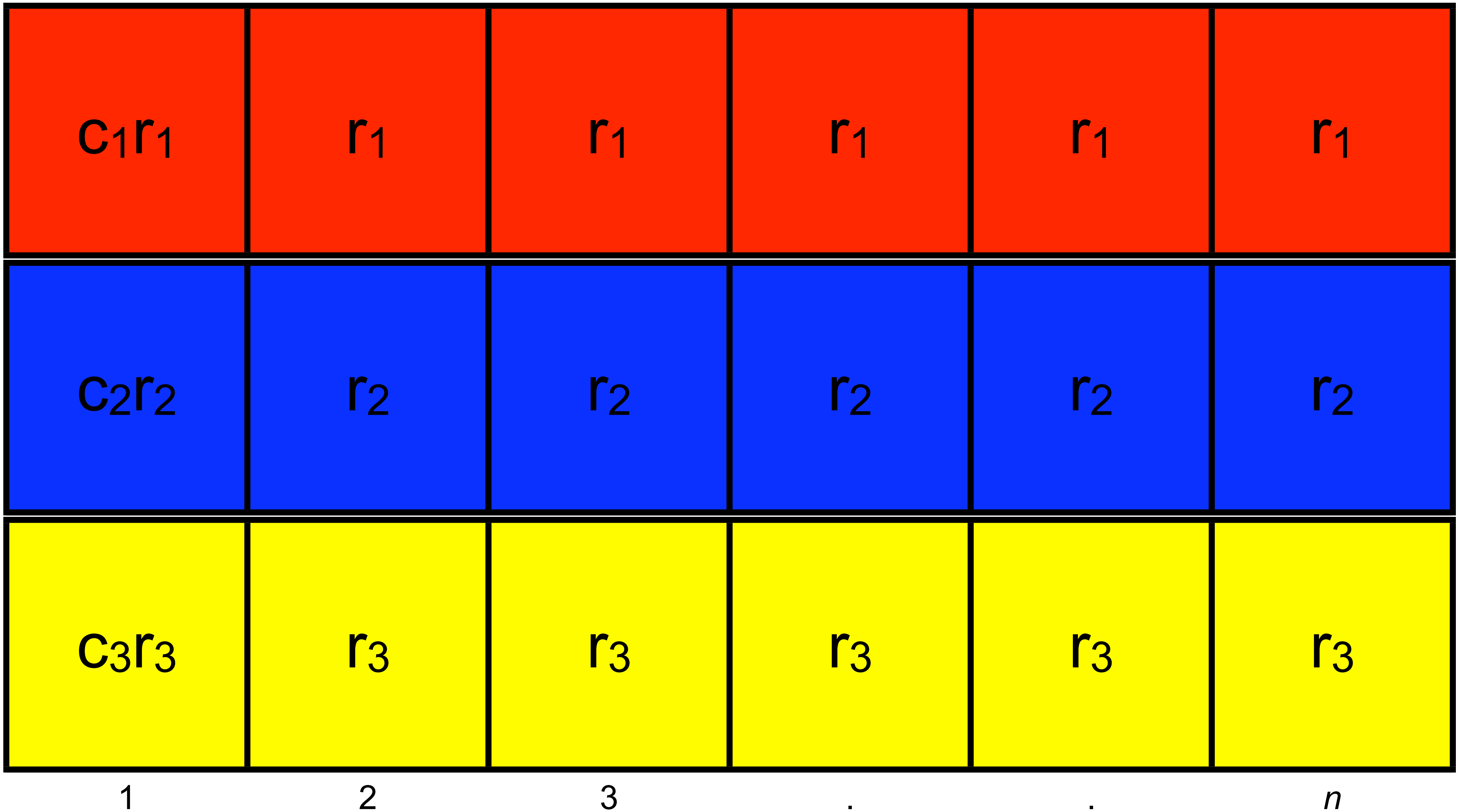

Just 3 pure tilings!

c_1r_1	r_1	r_1	r_1	r_1	r_1
c_2r_2	r_2	r_2	r_2	r_2	r_2
c_3r_3	r_3	r_3	r_3	r_3	r_3
1	2	3	.	.	*n*

total weight $W_n = c_1r_1^n + c_2r_2^n + c_3r_3^n$

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The same proof works for *k*th order recurrences with *k* distinct roots to its characteristic equation

Square weights: r_i $1 \le i \le k$

Domino weights: $-r_i r_j$ $1 \le i < j \le k$

t-omino weights: $(-1)^{t+1} r_{i_1} r_{i_2} \cdots r_{i_t}$

k-omino weights: $(-1)^{k+1} r_1 r_2 \cdots r_k$

The same proof works for kth order recurrences with k distinct roots to its characteristic equation

Initial Square weights: $c_i r_i$ $\qquad 1 \leq i \leq k$

Initial Domino weights:

$$-(c_i + c_j) r_i r_j \qquad 1 \leq i < j \leq k$$

Initial t-omino weights:

$$(-1)^{t+1}(c_{i_1} + c_{i_2} + \cdots + c_{i_t}) r_{i_1} r_{i_2} \cdots r_{i_t}$$

Initial k-omino weights:

$$(-1)^{k+1}(c_1 + c_2 + \cdots + c_k) r_1 r_2 \cdots r_k$$

What about Repeated Roots?

If h_n has characteristic equation of the form

$$(x - r)^2$$

then there exist constants c_1 and c_2 such that

$$h_n = c_1 r^n + c_2 n r^n$$

Repeated Single Root

If h_n has characteristic equation of the form

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then there exist constants $c_1, c_2, \ldots, c_k$ such that

$$h_n = c_1 r^n + c_2 n r^n + c_3 n^2 r^n + \ldots + c_k n^{k-1} r^n,$$

where $c_1, c_2, \ldots, c_k$ depend on the initial conditions.

General Situation

If h_n has kth degree characteristic equation of the form

$$(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_t)^{m_t}$$

then there exist constants c_{ij} where $1 \le i \le t$ and $1 \le j \le m_i$ such that

$$h_n = \sum_{i=1}^{t} \sum_{j=1}^{m_i} c_{ij} n^{j-1} r_i^n.$$

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This can also be proved combinatorially!

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where $c_1, c_2, \ldots, c_k$ depend on the initial conditions.

Tiling Model: Same as before, but with coins added!

If h_n has characteristic equation of the form

$$(x - r)^k$$

First, imagine the k roots are distinct with weights

$r_1, r_2, \ldots, r_k$.

Numerically, $r_1 = r_2 = \ldots = r_k = r$.

Tiles (including initial tiles) get the same weights as before. For t = 1, …, k

t-omino weights: $(-1)^{t+1} r_{i_1} r_{i_2} \cdots r_{i_t}$ or

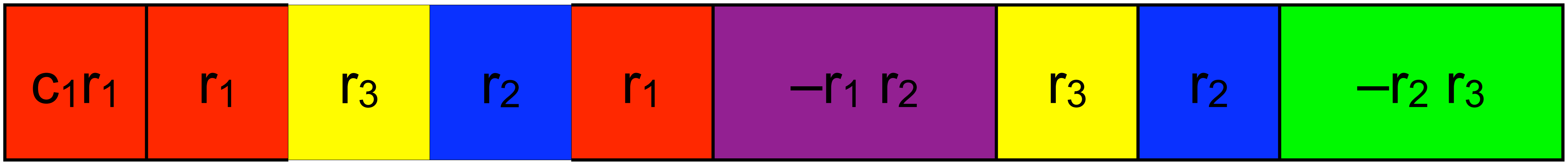
$$(-1)^{t+1}(c_{i_1} + c_{i_2} + \cdots + c_{i_t}) r_{i_1} r_{i_2} \cdots r_{i_t}$$

For a given tiling, let *m* denote the **largest index** to appear anywhere on the tiling.

Then place *m – 1* distinct coins on various tiles of the tiling (repetition allowed). **Coins may only be placed on tiles that contain r_m as a factor.**

Coins do not affect the weight of the tiling; they simply increase the number of tilings.

Examples of coined tilings



Here, *m = 3,* so we must place 2 distinct coins among the tiles containing r_3 -- the two yellow squares and green domino.(9 ways to coin the tiling)

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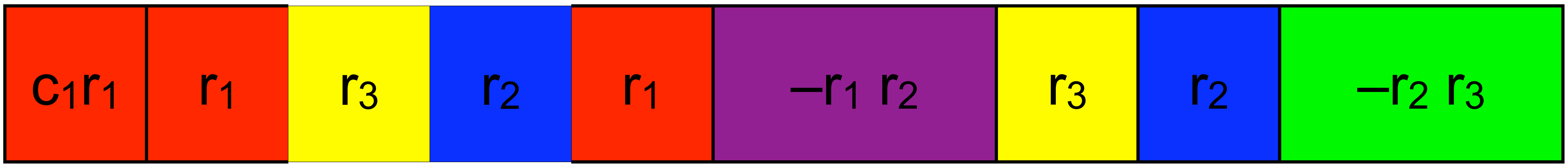

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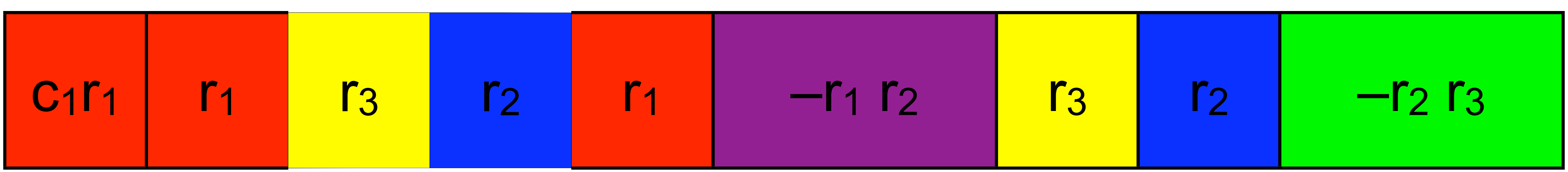

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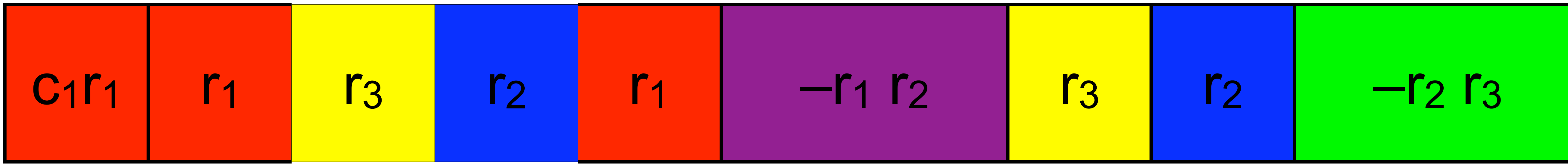

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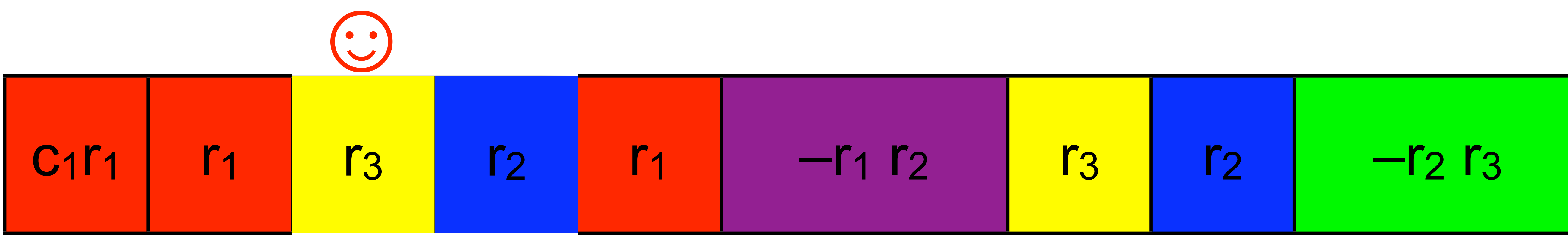

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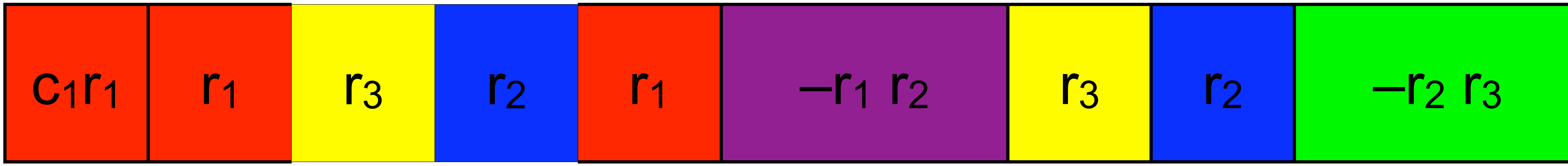

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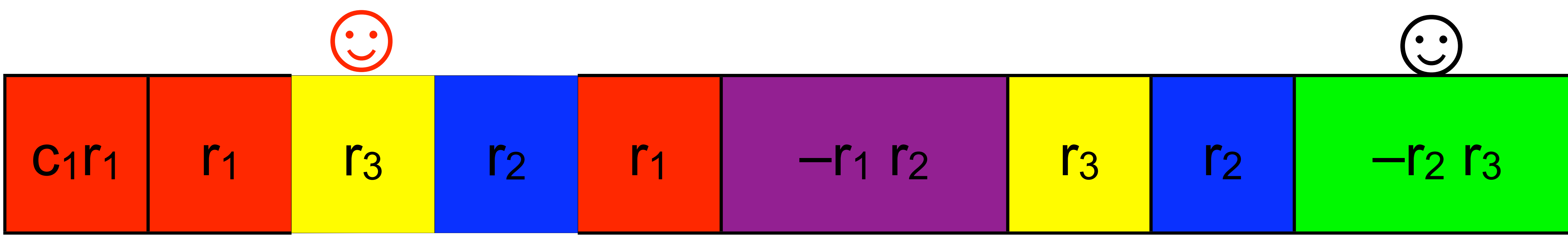

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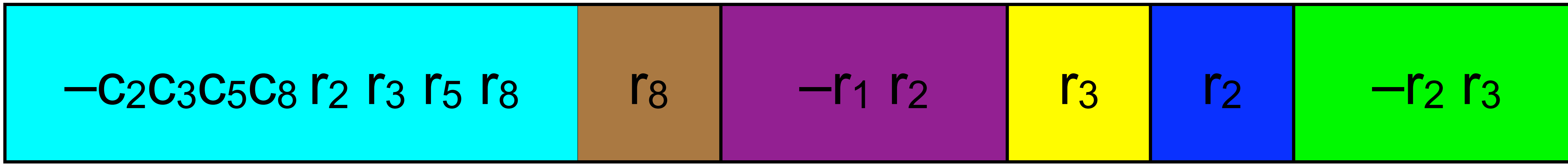

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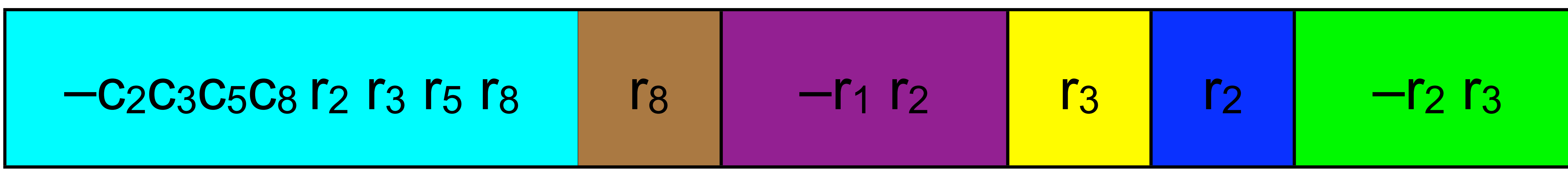

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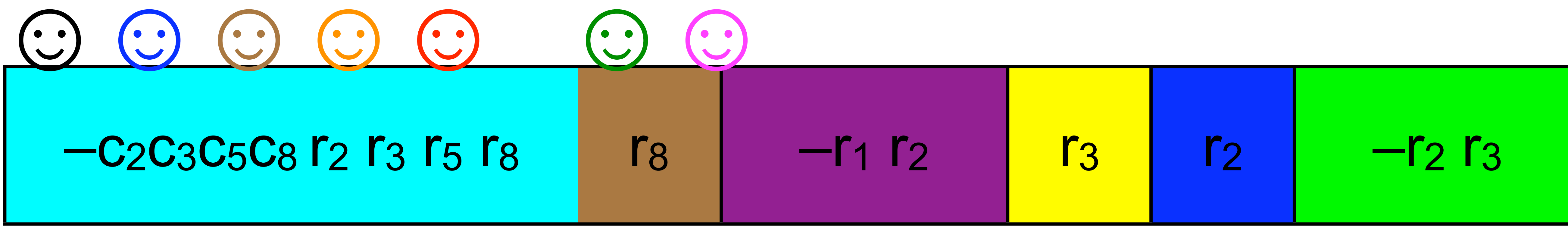

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Any tile of length greater than 1.

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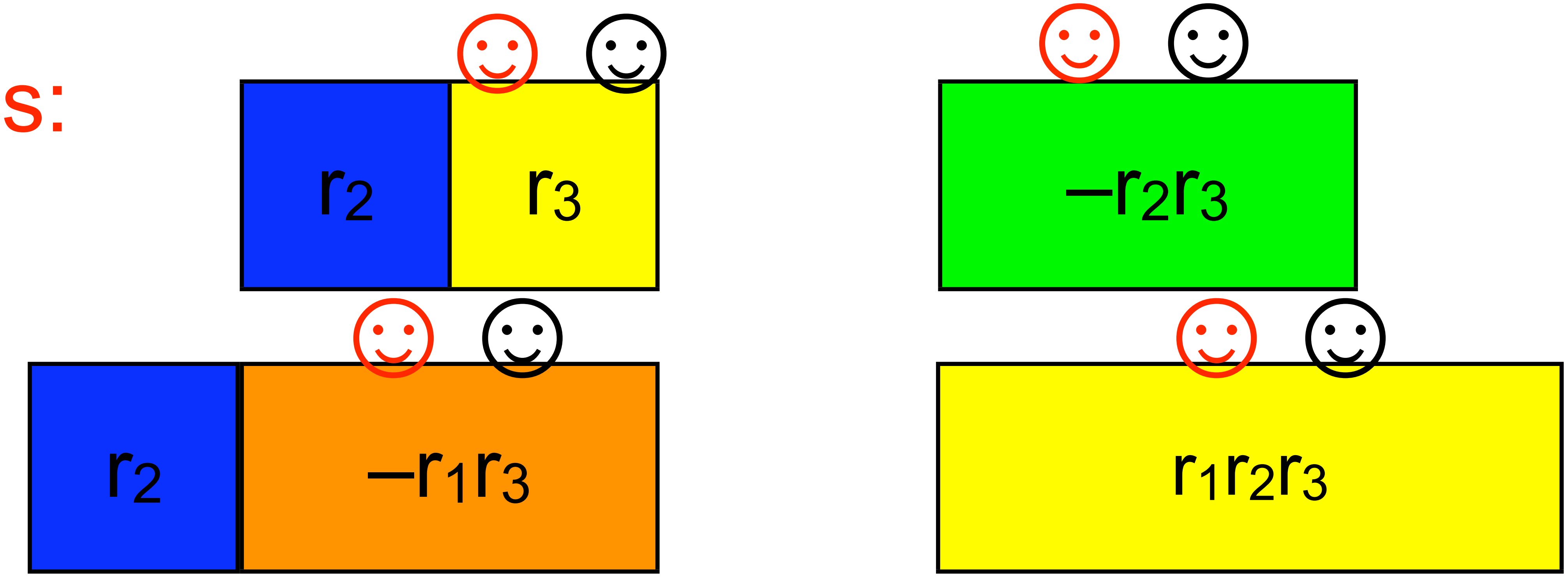

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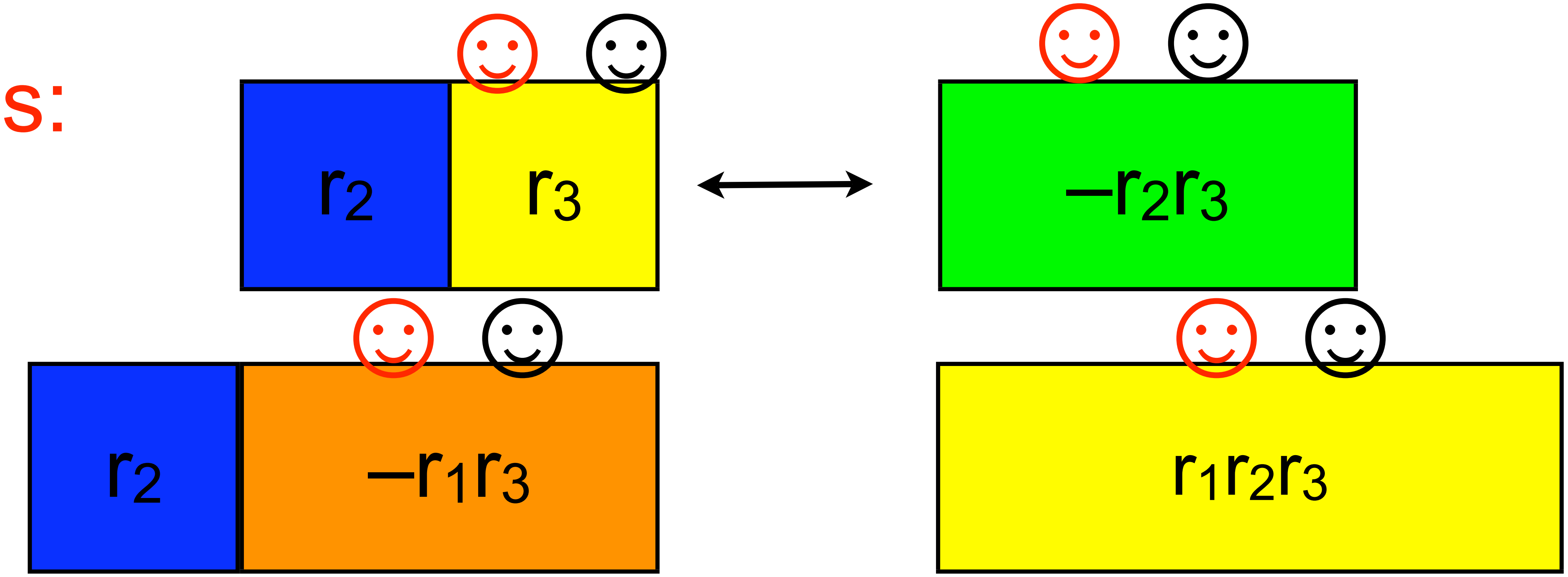

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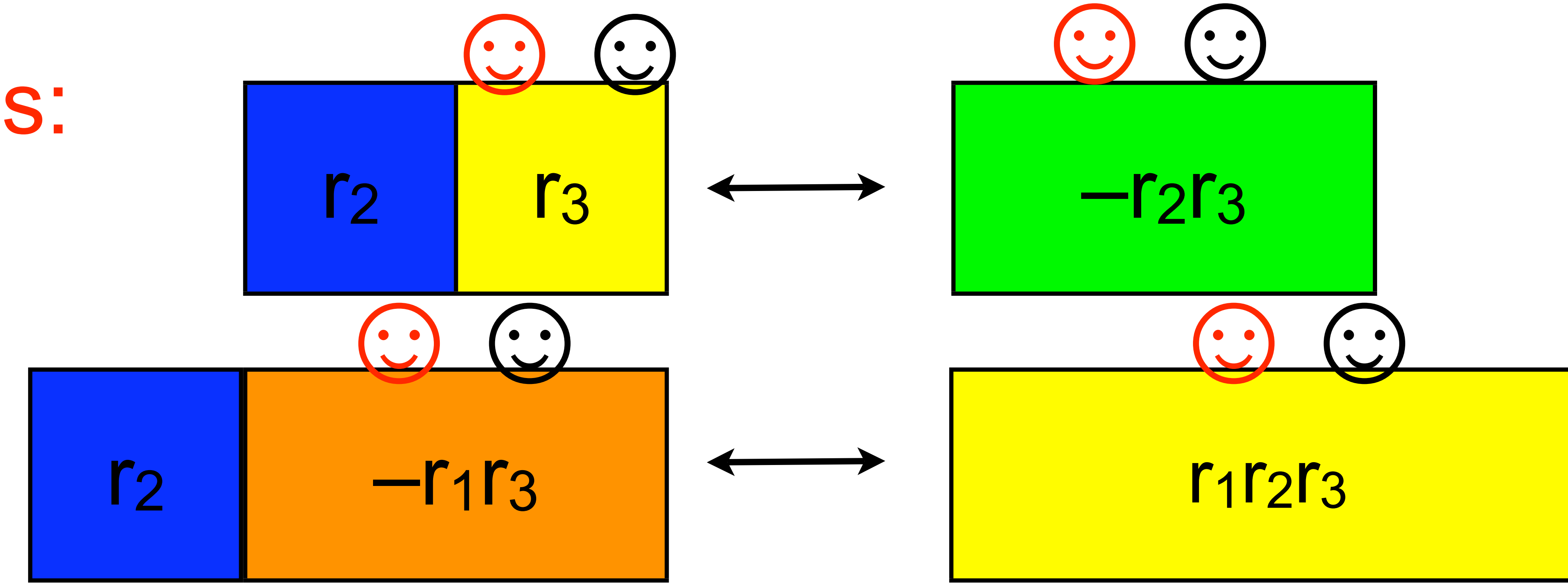

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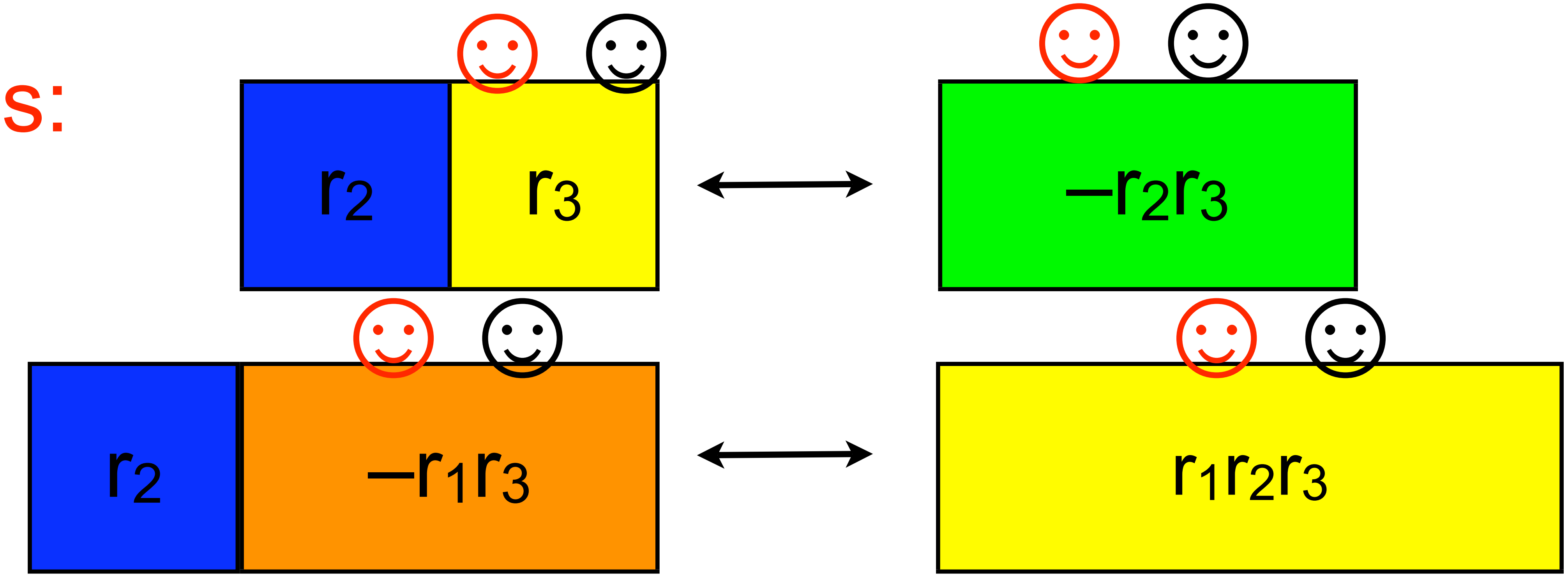


Non-initial impurities still sum to zero!

Coins follow the tile containing r_m

Initial impurities still sum to zero too!

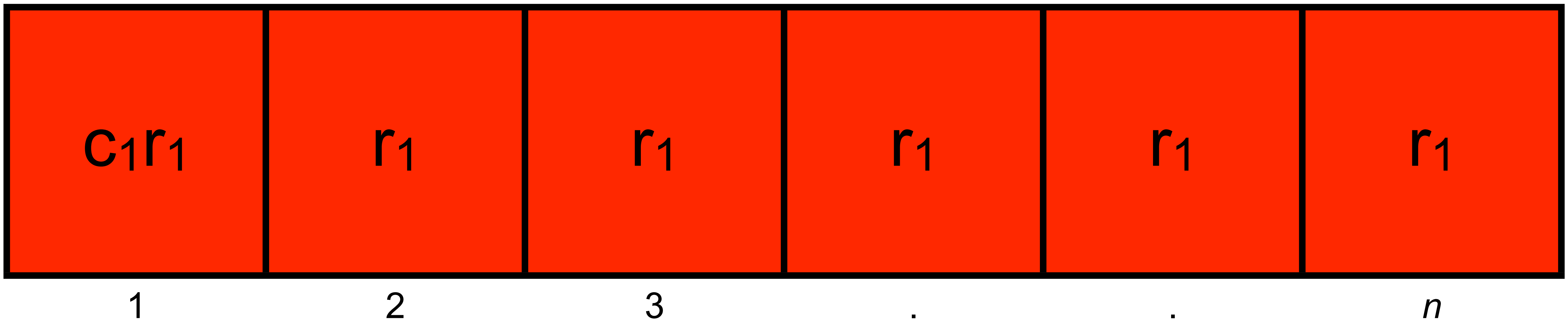
$$-(c_1+c_2)r_1r_2 \;+\; c_1r_1 \; r_2 \;+\; c_2r_2 \; r_1 \;=\; 0$$

$$(c_1+c_2+c_3)r_1r_2r_3 \;+\; c_1r_1 \; -r_2r_3 \;+$$

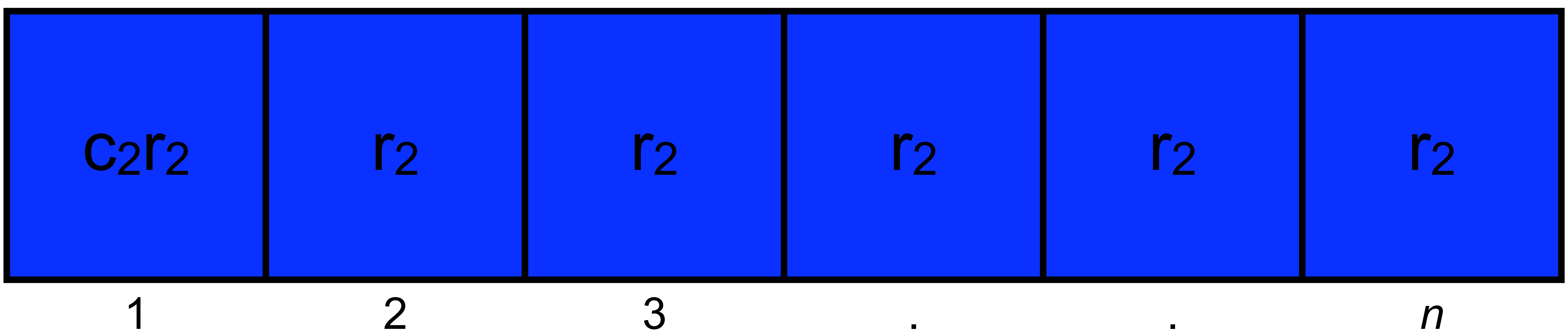
$$c_2r_2 \; -r_1r_3 \;+\; c_3r_3 \; -r_1r_2 \;=\; 0$$

Coins follow the tile containing r_m

The pure tilings



m = 1, so it has no coins, and weight = $c_1r_1^n$



m = 2, so it has 1 coin

It can be coined *n* ways, each with weight = $c_2r_2^n$

and...

The pure tilings

c_3r_3	r_3	r_3	r_3	r_3	r_3
1	2	3	.	.	*n*

m = 3, so it has 2 coins

It can be coined n^2 ways, each with weight = $c_2r_2^n$

and...

c_kr_k	r_k	r_k	r_k	r_k	r_k
1	2	3	.	.	*n*

m = k, so it has k-1 coins

It can be coined n^{k-1} ways, each with weight = $c_kr_k^n$

Let C_n be the total weight of all *coined* n-tilings.

Why does C_n satisfies the same recurrence as h_n?

This can be proved combinatorially by showing:

The total weight of tilings with any coins on the last tile is zero (by considering the *last* impurity).

So, by considering the last tile, we get *

$$C_n = a_1C_{n-1} + a_2C_{n-2} + \ldots + a_kC_{n-k}$$

(*Actually, this overcounts, but the total weight of the overcounted tilings is shown to be zero.)

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$$h_n = c_1 r_1^n + c_2 n\, r_2^n + c_3 n^2 r_3^n + \ldots + c_k n^{k-1} r_k^n$$

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☺

The general situation

If h_n has kth degree characteristic equation of the form

$$(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_t)^{m_t}$$

then there exist constants c_{ij} where $1 \le i \le t$ and $1 \le j \le m_i$ such that

$$h_n = \sum_{i=1}^{t} \sum_{j=1}^{m_i} c_{ij} n^{j-1} r_i^n .$$

where the values of c_{ij} depend on the initial conditions.

can be obtained from the last proof by just finding the first root that leads to an impurity.