BROWNIAN MOTION, MATH 642:592, Spring 2008

1. *Stochastic Processes in Continuous Time.*

A stochastic process in continuous time is a family $X = \{X(t); t \geq 0\}$ of random variables on a fixed probability space, indexed by all real $t \geq 0$. Given $\omega \in \Omega$, the function $t \to X(t)(\omega)$ is called the sample path of the process at ω. We can think of a stochastic process in this way as a random path.

In the continuous time framework, a filtration is a family $\mathbb{F} = \{\mathcal{F}_t; t \geq 0\}$ of σ-algebras that is increasing, in the sense that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $0 \leq s < t$. The *filtration* $\mathcal{F}^X$ *generated by a process* X is defined by $\mathcal{F}_t^X = \sigma\{X(s);\, 0 \leq s \leq t\}$, this being the smallest σ-algebra containing all events of the form $\{X(s) \in U\}$, where $s \leq t$ and U is a Borel subset of $\mathbb{R}$. More generally, we say that X is adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t\}$ if $\mathcal{F}_t^X \subset \mathcal{F}_t$ for all $t \geq 0$.

Given a filtration $\mathbb{F}$, a stochastic process X is an $\mathbb{F}$-martingale if (i) $\mathbb{E}\left[|X(t)|\right] < \infty$ for all $t \geq 0$, (ii) X is adapted to $\mathbb{F}$; and

$$\text{(iii)} \qquad \mathbb{E}\left[X(t) \mid \mathcal{F}_s\right] = X(s) \quad \text{a.s. for all } 0 \leq s < t. \tag{1}$$

Submartingales and supermartingales in continuous time are defined by replacing the equality in (1) by the suitable inequality. Notice that in defining martingales in discrete time we need only insist $\mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = X_n$ a.s., for all n, and it follows as a consequence that $\mathbb{E}\left[X_m \mid \mathcal{F}_n\right] = X_n$, whenever $m > n$. In continuous time, though, there is not a shortest step, and hence we must require (1) for all $0 \leq s < t$.

2. *The stationary, independent increment property.*

A stochastic process X is said to have *stationary, independent increments* if:

(i) for every $0 \leq s < t$, the increment $X(t) - X(s)$ is independent of $\mathcal{F}_s^X$;
namely, for each $A \in \mathcal{F}_s^X$ and each Borel subset $U \subset \mathbb{R}$, $\mathbb{P}\left(A \cap \{X(t) - X(s) \in U\}\right) = \mathbb{P}\left(A\right)\mathbb{P}\left(\{X(t) - X(s) \in U\}\right)$.;

(ii) for any $h > 0$, the distribution of $X(t+h) - X(t)$ is the same for all $t \geq 0$.

Property (i) alone is called the "independent increments property." Property (ii) is the "stationary increments property."

Stationary independent increment processes are the generalizations to continuous time of random walks of the form $X_n = \sum_1^n \xi_i$, where $\xi_1, \xi_2, \ldots$ is an sequence of independent, identically distributed random variables.

The following Lemma is left as an exercise using Monotone Class Theorems; for guidance, refer to RW, page 90 and page 116, or O, chapter 1.

Exercise 1. a) In order to show that X has the independent increment property it suffices to show that for every finite, increasing sequence of times $0 < t_1 < \cdots < t_n$, the random variables $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent.

b) If X has the independent increments property, then for every $s > 0$, the σ-algebras $\mathcal{F}_s^X$ and $\sigma\{X(t) - X(s)\,;\, t > s\}$ are independent.

The next exercise is also very basic and one we use repeatedly.

Exercise 2. If X has independent increments, if $X(t)$ is integrable for all t, and if $\mathbb{E}\left[X(t)-X(s)\right]=0$ for all $0 \leq s < t$, then X is a martingale.

3. *Definition of Brownian motion.*

Definition. A stochastic process W is called a standard Brownian motion if

(i) $W(0)=0$;

(ii) For all $t \geq 0$, $W(t)$ is square integrable, $\mathbb{E}\left[W(1)\right]=0$ and $\mathrm{Var}(W(1))=1$;

(iii) The sample paths of W are continuous, almost surely;

(iv) W has independent, stationary increments.

Notice that in this definition no conditions are imposed on the distribution of W. Basically, it just specifies an independent increments process with continuous paths. The properties that makes it "standard" are the normalizations $\mathbb{E}\left[W(1)\right]=0$ and $\mathrm{Var}(W(1))=1$. What is remarkable about this definition is that properties (i), (iii), and (iv) determine that all joint distributions of W are normal. This is the content of the next theorem. Note that it is more general than the theorem we stated and proved in class, because it does not assume *apriori* that W is integrable.

Theorem 1 *Let W satisfy properties (i), (iii), (iv) of the definition of Brownian motion. Then there are real numbers m and σ^2, where $\sigma^2 \geq 0$, such that for each $t \geq 0$, $W(t)$ is normal with mean mt and variance $\sigma^2 t$.*

Corollary 1 *Let W be a standard Brownian motion.*

(a) For every $t \geq 0$, $W(t)$ is normal with mean 0 and variance t.

(b) For any $0 < t_1 < \cdots < t_n$, the joint probability density of $(W(t_1), \ldots, W(t_n))$ is

$$p_n(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}} \frac{\exp\{-\frac{x_1^2}{2t_1} - \frac{(x_2-x_1)^2}{2(t_2-t_1)} - \cdots - \frac{(x_n-x_{n-1})^2}{2(t_n-t_{n-1})}\}}{\sqrt{t_1}\sqrt{t_2-t_1}\cdots\sqrt{t_n-t_{n-1}}}.$$

(c) W is a martingale and so also is $\{W^2(t)-t\}$.

Comments on proof. (a) is a direct corollary of Theorem 1 and condition (ii) of the definition of standard Brownian motion.

(b) is a consequence of the independence of increments and the fact, coming from stationarity of increments, that each increment $W(t_k)-W(t_{k-1})$ is normal with mean 0 and variance t_k-t_{k-1}.

(c) is a direct calculation using independence and normality of increments. $\diamond$

With these facts in hand, we can turn around and give a simple characterization of standard Brownian motion as a Gaussian process. A *Gaussian process* X is one such that the distribution of $(X(t_1), \ldots, X(t_n))$ is jointly Gaussian for every set of distinct times $0 \leq t_1 < t_2 < \cdots < t_n$.

Corollary 2 *Let W have continuous sample paths and assume that*

$$W \text{ is Gaussian, } \mathbb{E}[W(t)] = 0, \ \mathrm{Cov}(W(t), W(s)) = t \wedge s \ (\stackrel{\triangle}{=} \min(t,s)), \ \forall\, t, s \geq 0. \tag{2}$$

Then W is a standard Brownian motion.

Proof. If W is a Brownian motion, then it follows from part (b) of the previous Corollary that W is Gaussian. A Brownian motion has zero mean by definition. If $0 \leq s \leq t$, then $\mathbb{E}[W(t)W(s)] = \mathbb{E}[[W(t)-W(s)]W(s)] + \mathbb{E}[W^2(s)] = s$, because, since the increments of W are independent and have zero mean, $\mathbb{E}[[W(t)-W(s)]W(s)] = \mathbb{E}[W(t)-W(s)]\,\mathbb{E}[W(s)] = 0$.

Conversely, suppose that W is Gaussian, zero mean, and $\mathbb{E}[W(t)W(s)] = t \wedge s$. It first follows from this that for any $t \geq 0$, $\mathbb{E}[(W(t+h) - W(t))^2] = h$, implying that The increment $W(t+h) - W(t)$ is normal with mean 0 and variance h, independent of t. This proves stationarity of increments. Also, for any $0 \leq r < s < t$,

$$\mathbb{E}[(W(t) - W(s))(W(s) - W(r))] = t \wedge s - t \wedge r - s \wedge s + s \wedge r = s - r - s + r = 0.$$

This implies that if $0 \leq t_1 < t_2 < \cdots < t_n$, the covariance matrix of the random vector $(W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1}))$ is the identity matrix and hence that it is a vector of independent random variables. By exercise 1, this implies the independent increments property. $\diamond$

It should be noted that some authors define a standard Brownian motion as a mean zero Gaussian process satisfying (2) without insisting on path continuity. Then they prove that a continuous path version of the Brownian motion can be defined. The standard method, due to Kolmogorov, leads to the conclusion that for any given $\alpha < 1/2$, the paths of this version are Hölder continuous with Hölder exponent α, almost surely. We shall not treat this type of theorem in class. For a discussion, see RW, pages 59-65.

Later on we will prove Lévy's characterization of Brownian motion: any continuous process satisfying Lemma 1(c) and $W(0) = 0$ is a standard Brownian motion.

4. *The proof of Theorem 1.*

The rest of this lecture is devoted to a proof of Theorem 1. This will involve several preliminary steps, which we will carry out in greater generality than we need for Brownain motion, because we will use them in defining Lévy processes.

We will assume some background from the theory of characteristic functions. Here are the main relevant points. The characteristic function of a random variable X is $\phi_X(\lambda) \stackrel{\triangle}{=} \mathbb{E}\left[e^{i\lambda X}\right]$, $\lambda \in \mathbb{R}$. This is really the Fourier transform of the probability distribution F_x of X ($F_X(x) = \mathbb{P}(X \leq x)$):

$$\phi_X(\lambda) = \int_{\mathbb{R}} e^{i\lambda x}\, \mathrm{d}F_X(x).$$

The theory of characteristic functions gives us the following facts: (a) if $\phi_X = \phi_Y$, then $F_X = F_Y$; (b) the characteristic function of a normal random variable Z with mean m and variance σ^2 is

$e^{im\lambda-\lambda^2\sigma^2/2}$. The following bound on the tail probability of a random variable in terms of the characteristic is also fundamental:

$$\mathbb{P}\left(|X|>a\right) \leq \ell a \int_0^{1/a} \left(1-\mathrm{Re}(\phi_X(u))\right)\, du, \tag{3}$$

where $\ell \triangleq \sup_{|t|\geq 1} \frac{t}{t-\sin t} = (1-\sin(1))^{-1} \leq 6.5$.

Proof of (3): Since $\lim_{t\to 0} \sin t/t = 1$, let us agree to assign the value 1 to $\sin t/t$ when $t=0$; observe that for $t \neq 0$, $|\sin t/t| < 1$. Since $\mathrm{Re}(\phi_X(u)) = \mathbb{E}\left[\cos(uX)\right]$, and hence $1-\mathrm{Re}(\phi_X(u)) = \mathbb{E}\left[1-\cos(uX)\right]$, we have

$$\begin{aligned} a\int_0^{1/a} \mathbb{E}\left[1-\cos(uX)\right]\, \mathrm{d}u &= \mathbb{E}\left[1-\frac{\sin(X/a)}{X/a}\right] \geq \mathbb{E}\left[\mathbf{1}_{\{|X|>a\}}\left(1-\frac{\sin(X/a)}{X/a}\right)\right] \\ &\geq \inf_{|t|\geq 1}\left[\frac{t-\sin t}{t}\right]\mathbb{P}(|X|>a) \end{aligned}$$

To prove Theorem 1, we will show that if W satisfies conditions (i), (iii), and (iv) of the definition of Brownian motion, then $W(t)$ must have a characteristic function of the form $e^{imt\lambda-\lambda^2\sigma^2 t/2}$.

Part I. *Some facts about the characteristic functions of an independent increment process.*

In this section X shall denote a process with independent increments satisfying $X(0)=0$. It will not be assumed that the paths of X are continuous. However, it will be assumed that the sample paths of X are *right continuous*.

Here is one of the main observations from which the theory springs. Consider $X(1)$. For any positive integer n we can write $X(1)$ as the sum of increments of X over time intervals of length $1/n$:

$$X(1) = X(1/n) + [X(2/n)-X(1/n)] + [X(3/n)-X(2/n)] + \cdots + [X(1)-X(1-(1/n))].$$

But these increments are independent and identically distributed. If $M_n(\lambda)$ denotes the characteristic function of $X(1/n)$, it follows therefore that

$$\phi_{X(1)}(\lambda) = \mathbb{E}\left[\exp\{\lambda \sum_1^n X(k/n)-X((k-1)/n)\}\right] = M_n^n(\lambda). \tag{4}$$

In probability theory, a random variable which, for every positive integer n, is equal in distribution to a sum of n independent, identically distributed random variables, is called *infinitely divisible*. Since we can decompose $X(t)$ for any t in a similar manner, we see that $X(t)$ is infinitely divisible for all t.

Lemma 1 *(a) As $n\to\infty$, $M_n(\lambda)\to 1$ uniformly on $-K\leq\lambda\leq K$ for all $K>0$.*

(b) $\phi_{X(1)}(\lambda)\neq 0$ for all $\lambda\in\mathbb{R}$, and hence $M_n(\lambda)\neq 0$ for all λ and all $n\geq 1$.

Proof: (a) is a consequence of the right continuity of X, which implies that $X(1/n) \to X(0) = 0$ a.s. Fix a $K > 0$. For any $\epsilon > 0$, choose $\delta > 0$ so that if $|x| < \delta$, $|1 - e^{i\lambda x}| < \epsilon$ for all λ, $|\lambda| \leq K$. Then

$$\sup_{|\lambda| \leq K} \mid 1 - M_n(\lambda) \mid \leq \mathbb{E}\left[\mid 1 - e^{i\lambda X(1/n)} \mid\right] \leq \epsilon + 2\mathbb{E}\left[\mathbf{1}_{\{|X(1/n)| \geq \delta\}}\right].$$

Take $n \to \infty$. Then the second term converges to 0, by dominated convergence, and hence

$$\limsup_{n\to\infty} \sup_{|\lambda| \leq K} \mid 1 - M_n(\lambda) \mid \leq \epsilon.$$

But as $\epsilon > 0$ was arbitrary, we can let $\epsilon \downarrow 0$ and recover (a).

For any λ, $M_n(\lambda)$ will be non-zero for some n, by virtue of (a). Using this n, $\phi_{X(1)} = M_n^n(\lambda) \neq 0$, proving (b). $\diamond$

The characteristic function $\phi_Y(\lambda)$ of any random variable is continuous in λ, and $\phi_Y(0) = 1$. Therefore, using (b) of the previous lemma, there are unique, continuous, real-valued functions $\rho(\lambda)$ and $\theta(\lambda)$ such that $\rho(0) = 1$, $\theta(0) = 0$ and

$$\phi_{X(1)}(\lambda) = e^{\rho(\lambda) + i\theta(\lambda)}.$$

It follows also that for every $n \geq 1$, $M_n(\lambda) = e^{(1/n)(\rho(\lambda)+i\theta(\lambda))}$. We can then define $\ln \phi_{X(1)}(\lambda) \triangleq \rho(\lambda) + i\theta(\lambda)$, and similarly define $\ln M_n(\lambda)$, and write without ambiguity,

$$M_n(\lambda) = e^{(1/n)\ln \phi_{X(1)}(\lambda)}. \tag{5}$$

Lemma 2 $\quad \sup_{n\geq 1} n\mathbb{E}\left[X^2(\frac{1}{n})\mathbf{1}_{\{|X(1/n)|\leq 1\}}\right] < \infty.$

Remark. In class we proved Theorem 1 under the additional assumption (ii) and used the fact that $Var(W(1)) = 1$ to get this last lemma. Here we are deriving (c) even without assuming that $X(1)$ is integrable. Our proof will exploit crucially the right-continuity of the process. However, all the statements have generalizations to the situation in which $X(1)$ is replaced by any infinitely divisible random variable, although the proofs are a bit more delicate.

Proof: For $|x| \leq 1$, $\cos(x) \leq 1 - x^2/2 + x^4/24 \leq 1 - (11/24)x^2$. It follows that

$$n\mathbb{E}\left[X^2(\tfrac{1}{n})\mathbf{1}_{\{|X(1/n)|\leq 1\}}\right] \leq \frac{24}{11} n\mathbb{E}\left[1 - \cos(X(\tfrac{1}{n}))\right] = \frac{24}{11} n\left[1 - \mathrm{Re} M_n(1)\right]$$

From (5),

$$\lim_{n\to\infty} n[1 - M_n(1)] = \lim_{n\to\infty} \frac{1 - e^{(1/n)\ln \phi_{X(1)}(1)}}{1/n} = -\ln \phi_{X(1)}(1).$$

It follows that

$$\limsup_{n\to\infty} n\mathbb{E}\left[X^2(\tfrac{1}{n})\right] \leq -\frac{24}{11}\mathrm{Re}\left(\ln \phi_{X(1)}(1)\right) < \infty. \qquad \diamond$$

Part II. *Exploiting the continuity of paths.*

In this subsection, we derive the main consequence of the assumption of path continuity important to the proof of Theorem 1. The starting observation is that because of path continuity

$$\sup_{1\le k\le n} \mid X(\tfrac{k}{n}) - X(\tfrac{k-1}{n}) \mid \to 0, \qquad \text{a.s., and hence in probability, as } n\to\infty.$$

Now for every $\epsilon > 0$, the independent increment property implies that

$$\begin{aligned}
\mathbb{P}^n\left(\mid X(\tfrac{1}{n}) \mid < \epsilon\right) &= \Pi_1^n \mathbb{P}\left(\mid X(\tfrac{k}{n}) - X(\tfrac{k-1}{n}) \mid < \epsilon\right) \\
&= \mathbb{P}\left(\sup_{1\le k\le n} \mid X(\tfrac{k}{n}) - X(\tfrac{k-1}{n}) \mid < \epsilon\right) \\
&= 1 - \mathbb{P}\left(\sup_{1\le k\le n} \mid X(\tfrac{k}{n}) - X(\tfrac{k-1}{n}) \mid \ge \epsilon\right).
\end{aligned}$$

Taking logs, it follows that

$$0 = \lim_{n\to\infty} n \ln \mathbb{P}\left(\mid X(\tfrac{1}{n}) \mid < \epsilon\right) = \lim_{n\to\infty} n \ln\left[1 - \mathbb{P}\left(\mid X(\tfrac{1}{n}) \mid < \epsilon\right)\right].$$

The following is an immediate consequence.

Lemma 3 *Let X be an independent increment process satisfying properties (i), (iii), and (iv) in the definition of Brownian motion. Then*

$$\lim_{n\to\infty} n\mathbb{P}(|X(\tfrac{1}{n})| \ge \epsilon) = 0 \quad \textit{for every } \epsilon > 0.$$

Part III. *Completion of the proof.*

For convenience of notation, let $\phi(\lambda)$ denote $\phi_{X(1)}(\lambda)$. We will show that there is an m and σ^2 such that $\phi(\lambda) = \exp\{im\lambda - \sigma^2\lambda^2/2\}$. This will show that $X(1)$ is normal.

Start by writing,

$$\phi(\lambda) = \exp\{n \ln M_n(\lambda)\} = \exp\{(1+\epsilon_n(\lambda))n[M_n(\lambda) - 1]\} \tag{6}$$

The second equality should be interpreted as a definition of $\epsilon_n(\lambda)$. By Lemma 1 (a), for any $K > 0$

$$\epsilon_n(\lambda) = \frac{\ln(1+(M_n(\lambda)-1))}{M_n(\lambda)-1} - 1 \to 0 \quad \text{as } n\to\infty \text{ uniformly for } |\lambda| \le K. \tag{7}$$

(Define the first term on the right as 1 when $M_n(\lambda) = 1$).

Define

$$\phi(\lambda, x) = \begin{cases} \frac{e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x|\le 1\}}}{x^2 \wedge 1}, & \text{if } x \ne 0; \\ -\lambda^2/2, & \text{if } x = 0. \end{cases}$$

By Taylor's theorem with remainder bound, this function is bounded on $-K \le \lambda \le K$, $-\infty < x < \infty$, for all $K > 0$ and it is continuous on $-K \le \lambda \le K$, $-1 < x < 1$. Fix any λ. A simple calculation shows that

$$\begin{aligned} n[M_n(\lambda) - 1] &= n\mathbb{E}\left[e^{i\lambda X(1/n)} - 1\right] = \mathbb{E}\left[\phi(\lambda, X(1/n))[X^2(\tfrac{1}{n}) \wedge 1]\right] + i\lambda n\mathbb{E}\left[X(\tfrac{1}{n})\mathbf{1}_{\{|X(1/n)|\le 1\}}\right] \\ &= n\mathbb{E}\left[[\phi(\lambda, X(1/n)) + \frac{\lambda^2}{2}][X^2(\tfrac{1}{n}) \wedge 1]\right] + in\lambda\mathbb{E}\left[X(\tfrac{1}{n})\mathbf{1}_{\{|X(1/n)|\le 1\}}\right] \\ &\quad -\frac{\lambda^2}{2} n\mathbb{E}\left[X^2(\tfrac{1}{n}) \wedge 1\right]. \end{aligned} \tag{8}$$

We claim that the first term converges to 0 as $n \to \infty$. Indeed, if L denotes an upper bound on $|\phi(\lambda, x)|$,

$$\mid n\mathbb{E}\left[[\phi(\lambda, X(1/n)) + \frac{\lambda^2}{2}][X^2(\tfrac{1}{n}) \wedge 1]\right] \mid \le J(\delta) n\mathbb{E}\left[X^2(\tfrac{1}{n}) \wedge 1\right] + Ln\mathbb{P}\left(|X(\tfrac{1}{n})| \ge \delta\right). \tag{9}$$

where $J(\delta) = \sup_{|x|<\delta} \mid \phi(\lambda, x) + \lambda^2/2 \mid$. Now, as $n \to \infty$) the last term tends to 0 by Lemma 3. Since $\lim_{\delta\downarrow 0} J(\delta) = 0$ and since $\sup_n n\mathbb{E}\left[X^2(\frac{1}{n}) \wedge 1\right]$, by Lemma 2, we see that limit as $n \to \infty$ of the left-hand side in (9) is indeed 0.

Next, again using Lemma 2, there is a subsequence $\{n_k\}$ and a $\sigma^2 \ge 0$, such that

$$\lim_{k\to\infty} n_k\mathbb{E}\left[X^2(\tfrac{1}{n_k}) \wedge 1\right] = \sigma^2.$$

Therefore by letting $n_k \to \infty$ in (8), and by recalling (6) and (7),

$$\ln \phi(\lambda) = -\frac{\lambda^2}{2}\sigma^2 + i\lambda \lim_{k\to\infty} (1 + \epsilon_{n_k}(\lambda))\, n_k\mathbb{E}\left[X(\tfrac{1}{n_k})\mathbf{1}_{\{|X(1/n_k)|\le 1\}}\right] \tag{10}$$

The limit on the right must therefore exist, and it is clearly independent of λ. hence we have shown that for every λ there is an m such that

$$\phi_{X(1)}(\lambda) = \exp\{i\lambda m - \lambda^2\sigma^2/2\}.$$

It remains only to verify that for any $t \ge 0$, $\phi_{X(t)}(\lambda) = \exp\{i\lambda mt - \lambda^2 t\sigma^2/2\}$. But it follows from (5), that $\phi_{X(1/n)}(\lambda) = M_n(\lambda) = \exp\{i\lambda m/n - \lambda^1\sigma^2/2n\}$. For any k, $X(k/n)$ is a sum of k independent random variables with the same distribution as $X(k/n)$ and hence

$$\phi_{X(k/n)}(\lambda) = M_n^k(\lambda) = \exp\{i\lambda mk/n - \lambda^1\sigma^2 k/2n\}.$$

Thus $\phi_{X(t)}(\lambda) = \exp\{i\lambda mt - \lambda^2 t\sigma^2/2\}$ must be true for all rational $t \ge 0$. For any irrational t, let $s \downarrow t$ through rationals and use path continuity and dominated convergence to conclude that it holds for irrational t as well. This completes the proof of Theorem 1.

This lecture is based mostly on material in L. Breiman, *Probability*, Chapters 9 and 12.