Notes on the Dunford-Pettis compactness criterion

The Dunford-Pettis compactness criterion implies that uniform integrability is a necessary and sufficient condition for weak sequential compactness of a family of integrable random variables. The theorem applies to measure spaces more general than probability spaces, but we shall only discuss the probability space case.

In the discussion, $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a fixed probability space and L^1 for the space of integrable ($E[|X|] < \infty$) random variables on . Let $\{X_n\}$ be a sequence of random variables in L^1. Let X be in L^1, also. We say that $\{X_n\}$ converges weakly to X ($X_n \to X$ (weakly)) if

$$\lim_{n\to\infty} E\left[\xi X_n\right] = E\left[\xi X\right] \qquad \text{for every bounded random variable } \xi.$$

A family $\mathcal{X}$ of random variables in L^1 is said to be *(relatively) weakly sequentially compact* if any sequence contained in $\mathcal{X}$ contains a weakly convergent subsequence.

Remark: Recall from real analysis that the dual of L^1 is the set of all linear functionals ℓ on L^1 that are continuous in the sense that $E|X_n - X| \to 0$ implies $\ell(X_n \to \ell(X)$. This dual is isomorphic to the L^∞, the space of bounded random variables, by the identification of $\xi \in L^\infty$ with the linear functional $\ell_\xi(X) = E[\xi X]$. The weak topoolgy on L^1 is the smallest topology making all such linear functions continuous. So weak sequential compactness is a property concerning the weak topology.

Theorem 1 *A family $\mathcal{X}$ is weakly sequentially compact if and only if it is uniformly integrable.*

These notes will outline a proof of the fact that uniform integrability implies weak sequential compactness. The proof will employ the Radon-Nikodym theorem, and the fact that for any finite, positive K

$$\text{the product space } [-K,K]^\infty \text{ is compact in the product topology} \qquad (1)$$

To explain this, denote a typical element x of $[-K,K]^\infty$ by $x = (x_1, x_2, \ldots)$ and recall that a sequence $x^{(n)}$ of such elements converges in the product topology to $x \in [-K,K]^\infty$ if and only if $\lim_{n\to\infty} x_k^{(n)} = x_k$ for each $k = 1, 2, \ldots$; that is convergence in the product topology means component-wise convergence. We shall actually only need sequential compactness of $[-K,K]^\infty$, which, stated more directly than (1) means:

Lemma 1 *If $\{x^{(n)}\}$ is a sequence of elements of $[-K,K]^\infty$, then there exists a subsequence $\{x^{(n_k)}\}$ and an $x \in [-K,K]^\infty$¡ such that $x^{(n_k)}$ converges in the product topology to x.*

This result is a particular case of Tychonov's theorem, which in its greatest generality is proved using the Axiom of Choice—actually it's equivalent to the Axiom of Choice. But we can prove the statement here using only the compactness of $[-K, K]$ with respect to the usual topology on$\mathbb{R}$ and a diagonalization argument. By compactness of $[-K, K]$, there is a subsequence $n_1^1, n_2^1, \ldots$ such that $y_1 \triangleq \lim_{j\to\infty} x_1^{(n_j^1)}$ exists. By employing compactness of $[-K, K]$ again, there is a sub-subsequence $\{n_j^2\} \subset \{n_j^1\}$ such that $y_2 = \lim_{j\to\infty} x_2^{(n_j^2)} = y_2$ exists. Continuing in this manner, one obtains a sequence of subsequences $\{n_j^1\}, \{n_j^2\}, \ldots$ such that for each m, $\{n_j^m\}$ is a subsequence of $\{n_j^{m-1}\}$ and the limit $y_m = \lim_{j\to\infty} x_j^{(n_j^m)}$ exists. Define now the "diagonal" subsequence $n_k = n_k^k$, and $x = (y_1, y_2, \ldots)$. For every m, $n_m, n_{m+1}, \ldots$ is a subsequence of $\{n_j^m\}$. Hence $\lim_{k\to\infty} x_m^{n_k} = y_m$, for every m. $\diamond$

A standard construction from basic measure theory will also be used. The reader can verify the following. Let $\mathcal{C}$ be a class of subsets of Ω which includes the empty set, which is closed under finite intersections, and, finally, is such that if $A \in \mathcal{C}$, then its complement A^c is a finite disjoint union of elements of $\mathcal{C}$. Then the collection of finite disjoint unions of elements of $\mathcal{C}$ is an algebra of subsets of Ω, that is, it is closed under finite unions, finite intersections, and complementation and it contains Ω. We will apply this to the following example. Let $\{X_n\}$ be a sequence of random variables. Let $\mathcal{C}$ be the collection of finite intersections of events of the form $X_n^{-1}((s, r])$, where r and s are rational or infinite. Also, add the empty set to $\mathcal{C}$. It is immediate that $\mathcal{C}$ is closed under finite intersections. It is not hard to check that the complement of a set in $\mathcal{C}$ is a finite disjoint union of sets in $\mathcal{C}$. Therefore the collection $\mathcal{A}$ of finite disjoint unions of sets of $\mathcal{C}$ is an algebra. It is also clear that $\mathcal{C}$ is countable and so also is $\mathcal{A}$. Finally, it is also immediate that the smallest σ-algebra that contains $\mathcal{A}$ is the σ-algebra, $\sigma\{X_k; n \geq 1\}$, generated by $X_1, X_2, \ldots$; this is described by saying $\mathcal{A}$ generates $\sigma\{X_k; n \geq 1\}$.

Proof that uniform integrability implies relative weak sequential compactness.

It suffices to show that any uniformly integrable sequence $\{X_k\}$ contains a subsequence converging weakly in L^1. In the proof, we will write $E[X\mathbf{1}_A]$ as $E[X; A]$, for ease of notation. The proof proceeds in 3 main steps:

Step 1 Prove there is a subsequence k_n for which the limit $\lambda(A) \triangleq \lim_{n\to\infty} \mathbb{E}[X_{k_n}; A]$ exists for every A in $\sigma\{X_k; n \geq 1\}$, the σ-algebra generated by $\{X_k\}$.

Step 2 Prove that λ defined in Step 1 is a countably additive, bounded, signed measure on $\sigma\{X_k; n \geq 1\}$ that is absolutely continuous with respect to $\mathbb{P}$.

Step 3 Use the Radon-Nikodym theorem to conclude there is a $Y \in L^1(\Omega, \sigma\{X_k; n \geq 1\}, \mathbb{P})$ such that $d\lambda/d\mathbb{P} = Y$.

Step 4 Conclude from steps 1–3 that $X_{k_n} \to Y$ weakly.

Step 1, part 1. Let $\mathcal{A}$ denote the countable algebra of events consisting of finite disjoint union of intersections of events of the form $X_n^{-1}(r, q]$, where r and q are rational numbers, and $n \geq 1$. We constructed $\mathcal{A}$ above and we know that it generates the σ-algebra $\sigma\{X_n; n \geq 1\}$. We will first establish step 1 for the events of $\mathcal{A}$ and then show that λ extends to $\sigma\{X_n; n \geq 1\}$.

Let $A_1, A_2, \ldots$ be an enumeration of the sets of $\mathcal{C}$. Let

$$x^{(k)} \triangleq \left(\mathbb{E}\left[X_k; A_1\right], \mathbb{E}\left[X_k; A_2\right], \ldots\right)$$

Since $K = \sup_n \mathbb{E}\left[|X_n|\right] < \infty$, each $x^{(k)}$ defines an element of $[-K, K]^\infty$, and hence by Lemma 1 there is a subsequence $\{k_n\}$ such that

$$\lambda(A_m) \triangleq \lim_{n\to\infty} \mathbb{E}\left[X_{k_n}; A_m\right] \quad \text{exists for each } m \geq 1.$$

To simplify notation, set $Z_n = X_{k_n}$ so that we need not write out the subsequence explicitly.

Step 1, part 2. We show next that $\lambda(A) \triangleq \lim_{n\to\infty} \mathbb{E}\left[Z_n; A\right]$ is defined for all $A \in \sigma\{X_n; n \geq 1\}$. Here is where uniform integrability will be used fully. Let $\epsilon > 0$ be arbitrary. Because of uniform integrability, there exists a $\delta > 0$ such that $\mathbb{P}(B) < \delta$ implies that $\sup_n \mathbb{E}\left[|X_n|; B\right] < \epsilon$. Now, given any $A \in \sigma\{X_n; n \geq 1\}$, there is a $\tilde{A} \in \mathcal{C}$ such that $\mathbb{P}\left(A \triangle \tilde{A}\right) < \delta$. It follows that

$$\begin{aligned} \left|\mathbb{E}\left[Z_n; A\right] - \mathbb{E}\left[Z_m; A\right]\right| &\leq \left|\mathbb{E}\left[Z_n; \tilde{A}\right] - \mathbb{E}\left[Z_m; \tilde{A}\right]\right| + \mathbb{E}\left[|Z_n|; A \triangle \tilde{A}\right] + \mathbb{E}\left[|Z_m|; A \triangle \tilde{A}\right] \\ &\leq \left|\mathbb{E}\left[Z_n; \tilde{A}\right] - \mathbb{E}\left[Z_m; \tilde{A}\right]\right| + 2\epsilon. \end{aligned}$$

Taking $m, n \to \infty$ and then $\epsilon \downarrow 0$, we conclude that $\{\mathbb{E}\left[Z_n; A\right]\}$ is a Cauchy sequence, and thus that $\lambda(A) = \lim_{n\to\infty} \mathbb{E}\left[Z_n; A\right]$ is indeed well-defined.

Step 2. At this point we have constructed a set function λ on $\sigma\{X_n; n \geq 1\}$ and it is clearly finitely additive and bounded. We want to show that it is countably additive. This will be a consequence of the following statement of absolute continuity of λ with respect to $\mathbb{P}$: for every $\epsilon > 0$ there is a $\delta > 0$, such that $A \in \sigma\{X_n; n \geq 1\}$ and $\mathbb{P}(A) < \delta$ imply $\lambda(A) < \epsilon$. This is true because we know that uniform integrability imlies that for every $\epsilon > 0$ there is a $\delta > 0$, such that $\mathbb{P}(A) < \delta$ implies $\mathbb{E}\left[X_n; A\right] < \epsilon$ for all $n \geq 1$, and $\lambda(A)$ is defined as a limit of $\mathbb{E}\left[X_n; A\right]$. Now to derive countable additivity of λ, let $\{B_j\}$ be a countable disjoint sequence of events in $\sigma\{X_n; n \geq 1\}$. Since $\lim_{k\to\infty} \mathbb{P}\left(\cup_{j=k}^\infty B_j\right) = 0$, it follows immediately that $\lim_{k\to\infty} \lambda\left(\cup_{j=k}^\infty B_j\right) = 0$. By the finite additivity of λ,

$$\lambda\left(\cup_1^\infty B_j\right) = \sum_{j=1}^{k} \lambda(B_j) + \lambda\left(\cup_{j=k}^\infty B_j\right).$$

Now let $k \to \infty$ to obtain countable additivity.

Step 3. If $A \in \sigma\{X_n; n \geq 1\}$ and if $\mathbb{P}(A) = 0$, then $\mathbb{E}[X_n; A] = 0$ and hence $\lambda(A) = 0$. The Radon-Nidodym Theorem implies there exists a $Y \in L^1(\Omega, \sigma\{X_n; n \geq 1\}, \mathbb{P})$ such that $\lambda(A) = \mathbb{E}[Y; A]$ for all $A \in \sigma\{X_n; n \geq 1\}$.

Step 4. By definition of λ, it is certainly true that $\lim_{n\to\infty} \mathbb{E}[Z\xi] = \mathbb{E}[Y\xi]$ for every $\sigma\{X_n; n \geq 1\}$ measurable, simple function, these being the functions which are finite linear combinations of indicators of sets in $\sigma\{X_n; n \geq 1\}$. Every bounded, $\sigma\{X_n\}$-measurable ξ can be approximated uniformly by simple functions to an arbitrary degree of accuracy. Thus for any bounded random variable ξ.

$$\lim_{n\to\infty} \mathbb{E}[X_n\xi] = \lim_{n\to\infty} \mathbb{E}[X_n \mathbb{E}[\xi | \sigma\{X_n; n \geq 1\}]] = \mathbb{E}[Y \mathbb{E}[\xi | \sigma\{X_n; n \geq 1\}]] = \mathbb{E}[Y\xi] . \diamond$$

The proof that uniform integrability implies weak sequential compactness is complete. $\diamond$

The converse statement, that relative, weak sequential compactness implies uniform integrability, may be proved with the Vitali-Hahn-Saks theorem. One version of this theorem is as follows.

Theorem 2 *Let $\{\mu_n\}$ be a sequence of signed, countably additive measures on a σ-algebra $\mathcal{G}$ of subsets of a set Ω. Assume that $\sup_n |\mu_n| < \infty$, where $|\mu|$ denotes the total variation of μ. Assume that there is a countably additive positive measure ν such that $\mu_n \ll \nu$ for all n. Assume that*

$$\lambda(A) \triangleq \lim_{n\to\infty} \mu_n(A)$$

exists for all $A \in \mathcal{G}$. Then λ is countable additive and

$$\sup_n |\mu_n(A)| \to 0 \quad as \quad \nu(A) \to 0.$$

A general proof of this result using a uniform boundedness principle may be found in Dunford and Schwartz, *Linear Operators: Part I, General Theory*, Wiley Interscience, 1976, pp. 158-160. An "elementary proof" may be found in R. Ash, *Probability and Measure Theory.* The full Dunford-Pettis theorem is discussed and proved in the Dunford and Schwarz reference on pages 290-295.