1. History

This organization of the study of continued fractions can be traced to an article of Euler in 1737. The theorem that every quadratic irrational has a continued fraction that is eventually periodic was first proved by Lagrange in 1770. The version given by Gauss in 1801 remains the standard treatment of the subject, except for the characterization of inverse periods by Galois in 1828.

The book of Perron is the standard for both the arithmetic and analytic theory of continued fraction, and it includes references to many of the original articles.

Weil notes that these original sources are widely available in the collected works of their authors (although I did not find a copy of works of Lagrange in a search of the Rutgers library catalog), but he also gives a detailed description of many of the methods developed in the papers that he cites. In addition to the publications, there are letters that have been collected and published that give insight into the development of these results, and Weil includes these references as well.

Here are the references (including speculation, based on other citations, about where they may be found in collected works that have not been examined.

E. Galois, “Démonstration d’un théorème sur les fractions contiues périodiques”, Annales de mathématiques pures et appliques 19 (1828-29) (also in Œvres)
A. Weil, Number Theory: An approach through history; From Hammurapi to Legendre, Birkhäuser, 1984.

1. Reduction.

The modern definition of a reduced quadratic irrational \( \xi = (a + \sqrt{D})/c \) appears in Gauss’ book (Article 183). It requires \( \xi > 1 \) and the conjugate \( \xi^* = (a - \sqrt{D})/c \) to be between \(-1\) and \(0\).

The steps of the continued fraction of \( \xi \) are rational functions with rational coefficients, so their application to \( \xi^* \) will give the conjugate of their application to \( \xi \). These steps are determined to assure that all \( \xi_k > 1 \), so we need to show the \(-1 < \xi_k^* < 0\) for large enough \(k\). First note that, if this is true for some \(k\), it is also true for \(k + 1\). The continued fraction step involves subtracting \(a_k = [\xi_k]\), which moves \(\xi_k^*\) below \(-1\); and then inverting, which moves it back to the interval \((-1, 0)\).

Indeed, if there are at least two integers greater than \(\xi^*\) and less than \(\xi\), the next \(\xi^*\) will be in \((-1, 0)\) by the above argument.

If there is one integer greater than \(\xi^*\) and less than \(\xi\), when \(\xi\) is translated into \((0, 1)\), \(\xi^*\) falls in \((-1, 0)\). Then, \(-1, 0,\) and \(1\) are all between the next \(\xi\) and the next \(\xi^*\). As noted above, we get a reduced pair of conjugates in one more step.

If there is one integer between \(\xi\) and \(\xi^*\), but \(\xi < \xi^*\), one continued fraction step gives \(\xi^* < \xi\) with at least one integer between them. We have seen how these soon become reduced.

If there are no integers between \(\xi\) and \(\xi^*\), the continued fraction step translates both into \((0, 1)\) and then inverts. This increases the difference between the conjugates. Indeed, we have seen that there is an expansion by a scale factor greater than 1 in two such steps. Thus, a finite number of steps will be sure
to lead to conjugates whose difference is greater than 1. Two such numbers must have an integer between
them, so they soon become reduced.

We have shown that every quadratic irrational leads to a reduced pair of conjugates in a finite number
of steps of the continued fraction, and all subsequent steps give reduced pairs.

2. Reversing steps. If the pair \((\xi, \xi^*)\) is reduced, so is \((-1/\xi^*, -1/\xi)\) and if \(a = \lfloor -1/\xi^* \rfloor\),
\(-a - 1 < 1/\xi^* < -a\), but \(1/\xi^*\) is where the previous \(\xi^*\) wound up after subtracting \(\lfloor \xi \rfloor\). Since it would
have started in \((-1, 0)\) if the previous pair were reduced, this shows that \(a\) is also the greatest integer in
the previous \(\xi\) if that is reduced. This shows that every reduced pair appears in a purely periodic continued
fraction.

This investigation and its interpretation seems to have been noticed first by Galois. In addition to
showing that this definition of reduced pair is an exact characterization of numbers appearing in a period,
it shows that reversing the period gives the negative reciprocal of the conjugate.

This admits a neat description using the matrix interpretation of linear fractional transformations. The
number \(\xi\) is an abbreviation of the fraction \(\xi/1\), and the numerator and denominator of this fraction should
be considered as the components of a vector. The period of a continued fraction leads to a matrix

\[
P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}
\]

that takes this vector to a vector representing the same fraction. That is, to a vector in the same direction.
This means that this vector is an eigenvector of the period matrix. The corresponding eigenvalue is a
root of the characteristic polynomial of the matrix, which we know is \(x^2 - (a + d)x + (ad - bc)\). In
this case, the determinant \(ad - bc = (-1)^n\), so these eigenvalues are units of the ring with discriminant
\((a + d)^2 - 4(ad - bc) = (a - d)^2 + bc\). The equation whose roots are \(\xi\) and \(\xi^*\) is \(cx^2 + (d - a)x - b\),
which we recognize as having the same discriminant as the eigenvalue.

Since the factors of \(P\) are symmetric, the product in the reverse order is the transpose of \(P\). However, if

\[
U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

a direct calculation shows that \(UMU^{-1}\) is the transpose of the adjugate matrix of \(M\). For nonsingular
matrices, the adjugate represents the inverse linear fractional transformation, so transposing gives the inverse
action on the negative reciprocal of \(\xi\). The inverse of a unit \(\eta\) is \(\pm \eta^*\), so a conjugate must also be taken to
get the quantity belonging to the eigenvalue \(\eta\).