1. Let $a < b$ be given; let $f(x)$ be 3-times continuously differentiable on $[a, b]$, and suppose one knows the values $f(a)$, $f'(a)$, and $f(b)$.

1.a.: Find the interpolating polynomial $P(x) \in \mathcal{P}_3$ that interpolates $f(a)$, $f'(a)$, and $f(b)$. (Newton form is recommended. Note that the Hermite-Lagrange approach would produce a polynomial in $\mathcal{P}_3$ and would require information that we don’t have—and, in fact, are trying to approximate in what follows.)

1.b.: Suppose $P'(b)$ is used as an approximation of $f'(b)$. Use Taylor approximation of $f'(b)$ with base point $a$ and an appropriate error term (or any other correct method at your disposal) to show that the error bound

$$|f'(b) - P'(b)| \leq \frac{5}{6} M_3 \cdot h^2$$

is valid, where $h = b - a$ and

$$M_3 = \max \{|f^{(3)}(\xi)| : a \leq \xi \leq b \}.$$ 

(The constant $5/6$ is not the best possible: can you produce a better bound, and show that it is best possible?)

2. Consider the system of equations

$$x^3 - 3xy^2 + 1 = 0$$
$$3x^2y - y^3 = 0.$$

2.a.: Find an exact solution of these equations near $(0.6, 0.9)$ by solving the second equation for $y^2$ in terms of $x$, then substituting in the first equation, etc. (Square roots will be involved, but nothing worse.)

2.b.: Do two iterations of Newton’s method for systems as applied to this system, making the initial guess $(0.6, 0.9)$. Invert the “matrix derivative” of the system with as much accuracy as you can. Discuss whether your iterates appear to be converging to the exact solution you found in 2.a above.

3. Consider the equation $x^3 - x^2 - x - 1 = 0$, which has a unique real solution $\alpha \in [1, 2]$.

3.a.: A possible recasting of the problem of solving this equation as a fixed-point problem would be to rewrite it in the form

$$x = g(x) \equiv 1 + \frac{1}{x} + \frac{1}{x^2}.$$

3.b.: Find an interval $[a, b] \subseteq [1, 2]$ and a constant $\lambda < 1$ such that (1): if $x \in [a, b]$ then $g(x) \in [a, b]$ and (2): $|g'(x)| \leq \lambda$.

3.c.: Pick a point $x_0 \in [a, b]$. On the basis of your choice of the constant $\lambda$, how many iterations of $x_i = g(x_{i-1})$, beginning with your $x_0$, do you expect (on a theoretical basis) will be required to attain four correct decimal digits after the decimal point (absolute error $< 5 \times 10^{-5}$)?

3.d.: Do the computation and compare the result. (For the value of the root $\alpha$ correct to many places, see the class notes on Müller’s method.)

4. A family of methods (that we did not discuss in class) for finding roots of equations is given by the inverse interpolation methods. These represent the following approach to the problem of solving $f(x) = 0$ for a true root $\alpha$. Suppose that there is an interval $(a, b) \subseteq \mathbb{R}$ containing $\alpha$ and an interval $(c, d) \subseteq \mathbb{R}$ containing 0, such that $y = f(x)$ has an inverse $x = f^{-1}(y)$ defined for $c < y < d$ and taking its values in $(a, b)$. Suppose $\{x_0, \ldots, x_k\} \subseteq (a, b)$ are (known, previously-found [somehow]) approximations to $\alpha$. For each $x_j$, let $y_j = f(x_j) \neq 0$, and suppose each $y_j \in (c, d)$. Then even though $f^{-1}(y)$ has only a theoretical existence, one has enough data that one can explicitly produce a polynomial $P(y) \in \mathcal{P}_n$ such that $P(y_j) = x_j$ for $j = 0, \ldots, n$. Such a polynomial is called an inverse interpolant of $y = f(x)$, and the methods of root-finding we are developing here are the inverse interpolation methods.
4.a.: There is a value of $y$ which, when plugged into $P(y)$, can reasonably be expected to produce (as the value of $P(y)$) a better approximation to $\alpha$ than any of the $x_j$’s were. What value of $y$ would that be?

4.b.: (If you did not see how to attack 4.a, try it again after doing this section.) Suppose there are given two approximations $x_0, x_1$ to $\alpha$, with corresponding values $y_0, y_1$ for $y = f(x)$. Sketch the case in which $y = f(x)$ is a parabola passing through the $x$-axis at $(\alpha, 0)$ and sketch the graph of the inverse interpolant $x = P(y)$, where $P(y) \in P_1$ is obtained as discussed above. What method of root-finding have you “rediscovered”, and what was the answer to 4.a above in this case (of course the answer to 4.a is the same in all cases)?

4.c.: Suppose there are given three approximations $x_0, x_1, x_2$ to $\alpha$, with corresponding values $y_0, y_1, y_2$ for $y = f(x)$. The inverse interpolant $x = P(y)$ obtained as discussed above will in this case belong to $P_2$. Is its graph a parabola? Have you “rediscovered” Müller’s method? Why or why not? (Consider the case where the original $y = f(x)$ was a quadratic function, so its graph was a parabola.)

4.d.: Give a formula for estimating the error of the approximation developed in 4.c above in terms of derivatives of the function $f(x)$. You may assume that it and its inverse have derivatives of all orders, but your estimate should use the derivatives of $f(x)$. [The derivatives of $f^{-1}(y)$ can be expressed in terms of those of $f(x)$: for example, if $y_0 = f(x_0)$ then

$$\frac{d}{dy} f^{-1}(y_0) = \frac{1}{f'(x_0)}, \quad \frac{d^2}{dy^2} f^{-1}(y_0) = -\frac{f^{(2)}(y_0)}{[f'(x_0)]^3}, \text{ etc.}$$

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