1. Let \( f(x) = \frac{1}{1 + x^2} \), \( I = [-1, 1] \), \( N_1 = \{-1, -1/2, 0, 1/2, 1\} \), and \( N_2 = \{0, \pm \sqrt{5/8}\} \).

Find the values of each of the following at \( x = 3/4 \). (Exploit symmetry and don’t compute more than you have to!)

A. The 4th-degree polynomial interpolating \( f(x) \) on \( I \) with nodes \( N_1 \).

B. [Only if you are doing this on a computer] The 4th-degree polynomial interpolating \( f(x) \) on \( I \) with nodes \( N_2 \).

C. The piecewise-linear function interpolating \( f(x) \) on \( I \) with nodes \( N_1 \).

D. The piecewise-quadratic function defined on the mesh of width 1 that interpolates \( f(x) \) on \( N_1 \).

E. Compute \( f(3/4) \) and compare it with the results of (A–D) above to see which are the best/worst approximations.

2. Find the error term and use it to give a bound on the maximum error (worst-case bound) over \( I \) for the piecewise-linear approximation of \( 1.C \) above.

[To help your computation: \( f^{(3)}(x) = \frac{24x(1-x^2)}{(1+x^2)^4} \). (Can you see why this information is useful?) But you could have figured that out using MATLAB.]

3.A. Given a function \( f(x) \) defined on an interval \( I \) and two points \( a < b \) in that interval, determine (explicitly) the cubic polynomial \( p(x) \) that satisfies \( p(a) = f(a) \), \( p(b) = f(b) \), \( p'(a) = 0 = p'(b) \). {This is the sort of problem that the Newton form was designed to handle—use it.}

B. Suppose that \( f(x) \) is continuous and that \( a \) and \( b \) are so close that

\[ x, t \in [a, b] \Rightarrow |f(x) - f(t)| < \epsilon \]

Show that \( \max\{|f(x) - p(x)| : x \in [a, b]\} < \epsilon \), so “\( p(x) \) approximates \( f(x) \) uniformly within \( \epsilon \) on \([a, b]\).” {Caution: \( f(x) \) may have NO derivatives! so error-term estimates don’t help. This is slightly tricky but entirely elementary.}

4. Consider the problem of determining a 2nd-degree polynomial \( p_2(x) \) satisfying \( p_2(x_0) = f(x_0) \), \( p_2(x_2) = f(x_2) \) and \( p_2'(x_1) = f'(x_1) \), where \( x_0 < x_1 < x_2 \). Is it always possible to solve this problem for a given function \( f(x) \in C^3[x_0, x_2] \)? Is it possible to solve it for a restricted choice of the point \( x_1 \)? Discuss these questions, and give the solution when it exists (which may be “under all circumstances”).