Problem Set 11.

1. Find all integers \( x, y \) which solve the equation \( 3x^2 - 4y^2 = 11 \) by introducing a suitable order \( O \) in a number field and analyzing it.

2. Let \( K \subset L \) be distinct number fields
   a) Show that the units in the maximal order in \( K \) form a finite index subgroup of the units in the maximal order of \( L \) if and only if \( L \) has no real imbeddings (that is \( r_1(L) = 0 \), so that \( L \) is totally complex) , \([L : K] = 2\) and all imbeddings of \( K \) are real imbeddings (that is, \( K \) is totally real).
   b) We call a number field \( L \) a CM field if it contains a proper subfield \( K \subset L \) as in part a). Show that a number field \( L \) is a CM field if and only if there is a nontrivial field automorphism \( \rho \) of \( L \) such that for each imbedding \( \phi(x) \) we have \( \phi(\rho(x)) = \bar{\phi}(x) \). Show that the maximal totally real subfield of \( L \) is the fixed field of \( \rho \).
   c) Let \( L \) be a CM field, with automorphism \( \rho \) as in (b). Show that for any unit \( \epsilon \in O_L^* \), \( \rho(\epsilon)/\epsilon \) is a member of the group \( \mu_L \) of roots of unity in \( L \). Show that this induces a group homomorphism from \( O_L^*/\mu_L^*O_K^* \) to \( \mu_L^*/\mu_K^* \).
   d) Show that the homomorphism in (c) is injective, so that \([O_L^*: \mu_L O_K^*] \leq 2\).
   e) (Kummer) Show that \( L = \mathbb{Q}(\exp(2\pi i/p)) \) for \( p \) an odd prime is a CM field. Show that for such fields the index in (d) is equal to 1, hence show that every unit in the ring of integers in the cyclotomic field \( L \) is a root of unity times a unit in the maximal order of the maximal real subfield. Hint: Consider the field \( O_L/P \) where \( P \) is a prime in \( O_L \) dividing \( p \) and examine the action induced there by \( \rho \).
   f) Construct an example of a CM field \( L \) such that the index in (d) is 2. Hint: Consider a totally real field \( K \) and a unit \( u \) which is negative in all imbeddings, and form the field \( L \) by adjoining a square root of \( u \) to \( K \).

3. Suppose that \( L/\mathbb{Q} \) is a Galois extension with Galois group \( H \), the quaternion group of order 8 given by \( \{ \pm 1, \pm i \pm j, \pm k \} \) with \( \pm 1 \) in the center and \( ij = -ji = k, i^2 = j^2 = k^2 = -1 \).
   a) Show that \( L \) contains 3 quadratic fields, none of which are CM fields.
   b) Show that if \( L \) contains a nonreal element, then it is a CM field.