Week 7  

Wrapup of semisimple algebras, Brauer Group and Group representations

Jacobson II: 4.5-4.7, 5.1-5.3

1. The statements below will prove: Let $K$ be a field and let $D_1, D_2$ be division algebras over $K$ of dimensions $n_1, n_2$ (the centers may be larger than $K$). If $n_1, n_2$ are relatively prime, then $D_1 \otimes_K D_2$ is a division algebra. (a generalization of part of Jacobson II 4.6.9 from problem set 6)

a) Show that for any simple $K$-algebras $A_1, A_2$ there is a simple algebra $B$ and a surjective $K$-algebra homomorphism $A_1 \otimes K A_2 \rightarrow B$. (Hint: take the quotient by maximal two-sided ideal of $A_1 \otimes K A_2$).

b) For division algebras of finite dimension over $K$ consider an algebra $B$ constructed in a) as a vector space under $D_1 \otimes K 1$. Show that $\dim_K B = \dim_K D_1 \dim_K D_2 \leq \dim_K D_1 \dim_K D_2$.

c) If $\dim_K D_1$ is relatively prime to $\dim_K D_2$ show that $\dim_K B = \dim_K D_1 \dim_K D_2$ so that $D_1 \otimes_K D_2$ is a simple algebra.

d) Under the assumption of c), let $L$ be an irreducible ideal of the simple ring $A = D_1 \otimes_K D_2 = M_r(E)$, where the division algebra $E = (\text{End}_A(L))^{\text{opp}}$. Then $A = L^r$ as an $A$-module. $L$ is a module over $D1 \simeq D_1 \otimes 1$ and $D_2 \simeq 1 \otimes D_2$. Compute the dimension of $A$ as vector space over $D_1$ and $D_2$ and show that $r$ divides the dimensions of $D_1, D_2$ as vectors spaces over $K$. Conclude that if these dimensions are relatively prime, then $r = 1$ , establishing the statement at the beginning of this problem.

2. Let $G$ be a finite group and consider representations of $G$ on finite dimensional complex vector spaces.

a) Let $V, W$ be irreducible representations of $G$. Show that the decomposition of $V^* \otimes W$ into irreducible representations contains exactly 1 copy of the trivial representation of $G$ if $V, W$ are isomorphic, and 0 otherwise (see previously assigned Jacobson II 5.3.7).

b) Show that the element $t = \sum_{g \in G} g$ in $\mathbb{C}[G]$ is in the center of the group algebra, hence by Schur’s lemma acts by a scalar on any irreducible representation $V$. Show that if the subspace $W = tV \subset V$ is nonzero, then $V$ is the trivial representation. Show that $\chi_V(t) = 0$ if $V$ is not the trivial representation, and $\chi_V(t) = |G|$ if $V$ is the trivial representation.
c) Show using a), b) that if \( V, W \) are irreducible representations then \( \chi_{V \otimes W}(t) \) is 0 or \( |G| \) according as \( V, W \) are not isomorphic or are isomorphic. Deduce that the characters of irreducible representations of \( G \) form an orthonormal family of functions with respect to the pairing \( \langle \phi, \psi \rangle = (1/|G|) \sum_{g \in G} \phi(g^{-1}) \psi(g) \).

d) Show that if a representation \( V \) of \( G \) decomposes as a sum of irreducibles \( V = \bigoplus m_i V_i \) with \( V_i, V_j \) not isomorphic when \( i \neq j \), then \( \langle \chi_V, \chi_V \rangle = \sum m_i^2 \). Show that \( V \) is irreducible if and only if \( \langle \chi_V, \chi_V \rangle = 1 \).

e) Show that if \( W = \bigoplus m_i V_i \) is the decomposition of \( W \) into distinct irreducibles, then \( \langle \chi_W, \chi_{V_i} \rangle = m_i \). This, together with d) allows one to check from characters if a representation is irreducible, and if so how many times it appears in another representation.

3. The complex representations of \( S_4 \), the symmetric group on four letters.

a) Find the conjugacy classes in \( S_4 \), find all normal subgroups, and determine how many irreducible complex representations \( S_4 \) has.

b) Show that the group \( G \) of rotations preserving a cube permutes the 4 diagonals joining opposite corners of the cube and that this gives an injective group homomorphism from \( G \) to \( S_4 \). Show that \( G \) acts transitively on the 8 vertices of the cube and compute the stabilizer in \( G \) of a vertex. Show that the group \( G \) of rotations is isomorphic to \( S_4 \).

c) Show that the group \( G \) of rotations of the cube maps onto the group of permutations of the lines joining centers of opposite faces and that \( S_3 \) is a quotient of \( G \). Use this to find some irreducible representations of \( S_4 \) and to determine the dimension of the remaining irreducible representations.

d) Show that the natural representation of \( G \) as linear maps of \( \mathbb{R}^3 \) is irreducible.

e) Let \( W \) be the representation of \( G \) on the vector space of complex functions on the faces of the cube. Compute the character of \( W \) and express \( W \) as a sum of irreducible representations.

4. Let \( G \) be a subgroup of the group \( GL(n, \mathbb{C}) \) of invertible \( n \times n \) complex matrices. Suppose that \( \sum_{g \in G} \text{trace}(g) = 0 \). Show that \( \sum_{g \in G} g = 0 \).