1. Let $V$ be an infinite dimensional vector space over a division ring $D$ with countable basis $e_1, e_2, \ldots$. Let $R = \text{End}_D(V)$ be the ring of $F$-linear maps of $V$ to itself, with product the composition of maps.

   a) Show that the set $I \subset R$ of all linear maps with finite dimensional image is a two-sided ideal in $R$.

   b) Show that the only two sided ideals of $R$ are 0, $R$, $I$. Show that $I$ is a maximal two-sided ideal in $R$ and that the ring $R/I$ is a simple ring.

   c) Show that the set $I_j$ consisting of all endomorphisms vanishing on $e_n$ when $2^j$ divides $n$ is a left ideal of $R$. Show that the left ideals $I + I_j$ are distinct for distinct $j$ and form an increasing chain of left ideals and that $R/I$ is a simple ring which is not semisimple.

2. Let $K$ be a field, let $G$ be a group and let $K[G]$ be the group ring of $G$.

   a) Show that if kernel of the map $K[G] \rightarrow K$ which sends $\sum a_g g \in K[G]$ to $\sum a_g$ has a complement that it is a one dimensional $K$-vector space which is an eigenspace for the linear transformation multiplication by $g$ on $K[G]$.

   b) Show that when the characteristic of $K$ is a prime number $p$ and $K[G]$ is a semisimple ring then $G$ has no elements of order $p$.

   c) Show that the converse of Maschke’s theorem holds: The group ring $K[G]$ of a finite group $G$ is semisimple if and only if the order of $G$ is not divisible by the characteristic of $K$.

3. Let $R$ be a ring.

   a) Show that $R$ is semisimple if and only if every left $R$-module is projective.

   b) Show that $R$ is a division algebra if and only if every left $R$-module is free.

4. Show that if $R$ is a semisimple ring and $M$ is a finitely generated left $R$-module, then $\text{End}_R(M)$ is a semisimple ring.