Dimension of spaces of intertwining operators

Let $F$ be a field and let $V, W$ be finite dimensional vector spaces over $F$. Suppose that $L \in \text{End}_F(V), L' \in \text{End}_F(W)$. A vector space map $\phi : V \to W$ intertwines $L$ and $L'$ if and only if $\phi L = L' \phi$. The set of intertwining maps is a vector space $I_{L,L'}$ over $F$. For example if $V = W, L = L'$ then $I_{L,L'}$ is the vector space of endomorphisms of $V$ commuting with $L$.

We can recast the definition of $I_{L,L'}$ in terms of modules over a PID. Let $R = F[x]$ be the PID of polynomials in $x$ with coefficients in $F$. We can consider $V$ as an $R$–module by defining $xv = Lv$. Similarly $W$ is an $R$–module by defining $xv = L'v$. Then the vector space $I_{L,L'}$ equals $\text{Hom}_R(V,W)$, so it may be analyzed by module theoretic techniques.

Since $V, W$ are torsion modules over $R$ there exist polynomials $g_i, h_j \in R$ such that $V \simeq \oplus R/(g_i), W \simeq \oplus R/(h_j)$. By the fact that finite direct sums of modules are the same as finite products of that module and the defining properties of direct sums and products we have $\text{Hom}_R(V,W) \simeq \oplus_{i,j} \text{Hom}_R(R/(g_i), R/(h_j))$ so the problem is reduced to computing the vector space $\text{Hom}_R(R/(g_i), R/(h_j))$.

The homomorphisms of a cyclic module $R/g$ to any module $N$ are determined by the image of a generator, which can be taken to be any element of $n \in N$ such that $gn = 0$.

Lemma. Let $d$ be the greatest common divisor of $g_i, h_j$ so that $(g_i, h_j) = (d)$. Then $\text{Hom}_R(R/(g_i), R/(h_j)) \simeq (h_j/d)R/(h_j)$.

Proof: The elements of $n \in R/(h_j)$ such that $g_in = 0$ are represented by polynomials $p(x)$ modulo $h_j$ which satisfy that $p(x)g_i$ is a multiple of $h_j$. Those $p(x)$ are those which are a multiple of $h_j/d$.

Since $(h_j/d)R/(h_j)$ is the kernel of the quotient map from $R/(h_j)$ to $R/(h_j/d)$ it has dimension as a vector space over $F$ which equals $\deg(h_j) - \deg(h_j/d) = \deg(d)$. This shows that

$$\dim_F I_{L,L'} = \sum_{i,j} \deg \gcd(g_i, h_j)$$

In the special case that $V = W, L = L'$ we obtain a formula for the vector space of all linear maps commuting with $L$.

Theorem. Let $V$ be an $n$-dimensional $F$-vector space and let $L$ be an $F$-linear map of $V$ to itself. Then $\dim_F I_{L,L'} = \sum_1^n (2n - 2i + 1) \deg g_i$

Proof: The previous result gives that the dimension is $\sum_{i,j} \deg \gcd(g_i, g_j)$. Since $g_i$ divides $g_{i+1}$ this greatest common divisor is $g_{\min i,j}$. The sum $\sum_{i,j} \deg g_{\min i,j}$ equals $\sum_j (\sum_{i=1}^j \deg g_i + \sum_{i=j+1}^n \deg g_j)$ in which the term $\deg g_k$ appears $(n - k + 1) + (n - k)$ times.
For example, all polynomials in $L$ commute with $L$, forming a subspace of dimension equal to the degree of the minimal polynomial of $L$. The full space of commuting transformations exceeds this dimension by a positive amount if and only if some invariant factor $g_j$ for $j < n$ is not constant, that is if and only if the characteristic polynomial has degree greater than the minimal polynomial. Hence the characteristic polynomial of $L$ equals the minimal polynomial if and only if any transformation commuting with $L$ is a polynomial in $L$.

For a second example we determine the matrices commuting with the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 2
\end{pmatrix}.
$$

Since this matrix is in rational canonical form the invariant polynomials are $g_1 = x - 1, g_2 = (x - 1)^2$ so that the dimension of the space of matrices commuting with this one is $(6-4+1) + (6-6+1)2 = 5$. The proof of the theorem above shows that the linear transformations commuting with the transformation defined by the matrix above are $F[x]$-module endomorphisms of $F[x]/(x - 1) \oplus F[x]/(x - 1)^2$. Using the basis $1; 1, x$ we see that a module map must map $1 \in F[x]/(x - 1)$ to an element of form $(a, b(x - 1))$, and $1 \in F[x]/(x - 1)^2$ to an element of the form $(c, d + ex)$ and hence $x$ to $(c, -e + (d + 2e)x)$. Thus the matrices commuting with the matrix above are those of the form

$$
\begin{pmatrix}
a & -b & b \\
c & d & e \\
c & -e & d + 2e
\end{pmatrix}.
$$