Math 551, Assignment 7, due Monday, December 2, in my mailbox

Throughout this assignment, $V$ is a finite-dimensional vector space over the field $k$, and $B = \{v_1, \ldots, v_n\}$ is an ordered basis of $V$.

1. Let $\mathcal{B}$ be a bilinear form on $V$. Let $\beta_L, \beta_R$ be the associated mappings from $V$ to $V^*$. Show that
   a) $\beta_L = \beta_R^*$
   b) $[\beta_R]^B_{\mathcal{B}} = \{\mathcal{B}\}_B$. (Or is it $[\beta_L]^B_{\mathcal{B}}$?)
   c) The set of all matrices of the form $\{\mathcal{B}\}_B$, as $B$ varies over all the ordered bases of $V$ (but $B$ remains fixed) is an equivalence class under the equivalence relation $\approx$ defined by $A \approx A'$ if and only if $A' = P^T A P$ for some invertible $P$.

2. Suppose that $T : V \to V$ is a diagonalizable linear transformation. Let $W \subseteq V$ be a subspace such that $T(W) \subseteq W$. Thus $T$ induces linear transformations $T|W : W \to W$ and $T|V/W : V/W \to V/W$, which you are asked to prove are both diagonalizable as well. (Hint. Relate their minimal polynomials to $\mu_T$.)

3. Suppose that $T : V \to V$ and $U : V \to V$ are linear transformations such that
   $$TU = UT.$$ 
   a) Show that for any eigenspace $W$ of $T$, or indeed for any generalized eigenspace $W$ of $T$, $U(W) \subseteq W$.
   b) Suppose that $T$ and $U$ are both diagonalizable (that is, there is a basis of $V$ whose elements are eigenvectors for $T$, and there is also a basis whose elements are eigenvectors for $U$). Show that they are simultaneously diagonalizable, that is, there is a basis of $V$ whose elements are eigenvectors both for $T$ and for $U$. (Use the previous problem.)
   c) Interpret b) as a theorem about matrices.

4. A lattice $\Lambda$ in $\mathbb{R}^n$ is defined to be an abelian subgroup of $\mathbb{R}^n$ (under addition) which is generated by an $\mathbb{R}$-basis $B$ of $\mathbb{R}^n$.
   a) Show that $\Lambda$ is then a free abelian group with basis $B = \{v_1, \ldots, v_n\}$.

   The lattice $\Lambda$ is said to be integral if and only if $(v, w) \in \mathbb{Z}$ for all $v, w \in \Lambda$. Here $(\cdot, \cdot)$ is the standard Euclidean inner product. The lattice dual of $\Lambda$ is defined to be the group
   $$\Lambda^* = \{v \in \mathbb{R}^n \mid (v, w) \in \mathbb{Z} \text{ for all } w \in \Lambda\}.$$ 
   b) If $\Lambda$ is integral, show that $\Lambda \subseteq \Lambda^*$, that
   $$|\Lambda^*/\Lambda| = |\det((v_i, v_j))_{i,j=1}^n|,$$
   and that the structure of $\Lambda^*/\Lambda$ can be revealed by applying suitable integer row and column operations to the matrix $[(v_i, v_j)]_{i,j=1}^n$.

   (Hint. Let $w_1, \ldots, w_n \in \mathbb{R}^n$ be such that $(v_i, w_j) = \delta_{ij}$ (why do these vectors exist?) and show that $\Lambda^*$ is a lattice with basis $\{w_1, \ldots, w_n\}$. Then see the previous assignment.)
   c) What is the structure of the group $\Lambda^*/\Lambda$ in the following cases?
      (i) $B = \{[1 \quad -1 \quad 0], [0 \quad 1 \quad -1]\}$
      (ii) $B = \{[1 \quad -1 \quad 0], [0 \quad 1 \quad -1], [0 \quad 0 \quad 1 \quad -1], [0 \quad 0 \quad 1 \quad 1]\}$. 