As usual, \( G \) is a group.

1. Prove the following two statements using group actions.
   a) Suppose that \( H \leq G \) and \( |G : H| = n < \infty \). Show that there is a subgroup \( K \leq H \) such that \( K \triangleleft G \) and \( |G : K| \) divides \( n! \), and in particular \( |G : K| < \infty \).
   b) Suppose that \( H_1 \leq G \) and \( H_2 \leq G \) are two subgroups of finite index in \( G \). Show that \( H_1 \cap H_2 \) has finite index in \( G \). (Hint. Use an action on the disjoint union \((G/H_1) \cup (G/H_2)\).)

2. Let \( \Omega \) be an infinite set and define
   \[
   \Sigma_\Omega^n = \{ \sigma \in \Sigma_\Omega \mid \sigma(\omega) = \omega \text{ for all but finitely many } \omega \in \Omega \}.
   \]
   a) Show that \( \Sigma_\Omega^n \triangleleft \Sigma_\Omega \).
   b) Make a reasonable definition of \( A_\Omega^n \), and show that \( A_\Omega^n \) is simple.

3. Suppose that \( G = \langle S \rangle \), and let \( X \) be a subset of \( G \). Show that \( X = G \) if and only if \((S \cup S^{-1})X \subseteq X\), where \( S^{-1} = \{ s^{-1} \mid s \in S \}\).

4. Let \( G = gp(s_1, \ldots, s_n | R) \), where \( R \) is the set of all relators of the form
   \[
   s_i^2, 1 \leq i \leq n
   \]
   \[
   (s_is_j)^2, 1 \leq i < j \leq n, j > i + 1
   \]
   \[
   (s_is_{i+1})^3, 1 \leq i < n
   \]
   Show that \( G \cong \Sigma_{n+1} \). (Hints. Map \( s_i \mapsto (i+1) \in \Sigma_{n+1} \). Also, let \( H \) be the subgroup of \( G \) generated by \( s_1, \ldots, s_{n-1} \), and show that \( G = H \cup s_nH \cup s_{n-1}s_nH \cup \cdots \cup s_1s_2 \cdots s_nH \).)

5. Each of the following statements is false. Give a counterexample in each case. Also alter the underlined part of the statement in as reasonable way as possible so that the statement becomes true; then prove the altered statement.
   a) If \( \phi : G \rightarrow H \) is any homomorphism and \( L \) and \( M \) are any subgroups of \( G \), then \( \phi(L \cap M) = \phi(L) \cap \phi(M) \).
   b) If \( K \operatorname{char} G \) and \( K \leq L \leq G \), then \( L \operatorname{char} G \) if and only if \( L/K \operatorname{char} G/K \).
   c) If \( P \in \text{Syl}_p(G) \) and \( H \) is any subgroup of \( G \), then \( P \cap H \in \text{Syl}_p(H) \).
   d) If \( G \) possesses a composition series and \( H \) is any subgroup of \( G \), then \( H \) possesses a composition series.
   e) If \( A, B \text{ and } C \) are subgroups of \( G \) such that \( A \triangleleft B \), then \( A \cap C \triangleleft B \cap C \) and \( B \cap C/A \cap C \cong B/A \).

6. If \( k \) is any field and \( n \) is any natural number, then we define \( Z = \{ cI \mid c \in k^\times \} \), \( \text{PGL}_n(k) = \text{GL}_n(k)/Z \) and \( \text{PSL}_n(k) = \text{SL}_n(k)/Z \cap \text{SL}_n(k) \). Show that \( \text{PGL}_n(k) \) has a normal subgroup \( H \cong \text{PSL}_n(k) \), and that \( \text{PGL}_n(k)/H \cong k^\times/(k^\times)^n \).

7. Suppose that \( G \) is a finite solvable group. Show that if \( x_1, \ldots, x_r \in G \) are elements of orders \( p_1^{a_1}, \ldots, p_r^{a_r} \), respectively, where \( p_1, \ldots, p_r \) are distinct primes and \( a_1, \ldots, a_r \) are nonnegative integers, and if \( x_1x_2 \cdots x_r = 1 \), then \( a_1 = a_2 = \cdots = a_r = 0 \). (Hint. Use induction. Remark. It is also true that any finite group with this property is solvable, but this is a very deep theorem.)

8. Show that up to isomorphism, there are exactly three nonabelian groups of order 12: \( A_4 \), \( Z_2 \times \Sigma_3 \) and \( gp\langle x, y \mid x^4, y^3, xyx^{-1}y \rangle \).