1. An action of a group $G$ on a set $\Omega$ is said to be \textit{doubly transitive}, or $2$-transitive, if and only if for every $(\alpha, \beta)$ and $(\gamma, \delta) \in \Omega \times \Omega$ such that $\alpha \neq \beta$ and $\gamma \neq \delta$, there exists $g \in G$ such that $g\alpha = \gamma$ and $g\beta = \delta$.

   a) Show that if $|\Omega| > 2$, then $G$ is doubly transitive on $\Omega$ if and only if $G$ is transitive on $\Omega$ and also $G_\alpha$ is transitive on $\Omega - \{\alpha\}$ for some $\alpha \in \Omega$.

   b) Let $n$ be an integer such that $n > 1$. Show that the natural action of $SL_n(C)$ on $C^n - \{0\}$ (i.e. the set of all nonzero $n \times 1$ column vectors) is transitive but not doubly transitive, while the natural action of $SL_n(C)$ on the set of 1-dimensional subspaces of $C^n$ is doubly-transitive.

2. The quaternion group $Q_8$ can be defined as the subgroup $\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \rangle$ of $SL_2(C)$. Demonstrate the following: a) $|Q_8| = 8$; b) $Q_8$ contains a unique element of order 2; c) Every subgroup of $Q_8$ other than $Q_8$ itself is cyclic; d) $Q_8 \not\cong D_8$.

3. Let $G$ be a group of order 264. What can you say about the number of elements of order 11 in $G$? Can these elements all be conjugate in $G$?

4. Show that if $p$ is a prime divisor of the finite group $G$, then the number of subgroups of $G$ of order $p$ is congruent to 1 mod $p$. (Hint. Begin by adapting the proof of the corresponding part of Sylow’s Theorem.)

5. Let $G = \Sigma_4$. Write down representatives of the conjugacy classes of $G$ (i.e. exactly one element from each conjugacy class). For each representative $g$ which you have written, compute $|C_G(g)|$ and give generators of $C_G(g)$. Do the same for $A_4$. (The counting principle is useful here!)

6. a) Let $H$ be any subgroup of $\Sigma_n$. Show that $|H : H \cap A_n| = 1 \text{ or } 2$.

   b) Let $g \in A_n$. Show that each $\Sigma_n$-conjugate of $g$ (i.e. each permutation of the same cycle shape as $g$) is already conjugate to $g$ by an element of $A_n$ if and only if $g$ centralizes some odd permutation in $\Sigma_n$.

   c) Give an example of an element $g \in A_{15}$ for which the conditions in b) are false.

7. Suppose that $g, h \in \Sigma_n$ and $gh = hg$.

   a) Show that if $\Omega_1$ is an orbit of $\langle g \rangle$ on $\{1, 2, \ldots, n\}$, then $h\Omega_1$ is also such an orbit.

   b) Express the orbits of $C_{\Sigma_n}(g)$ in terms of the cycle decomposition of $g$. 

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Math 551, Assignment 2

Due Thursday, Sept. 28 in class