Computing the Jordan Canonical Form

Let $A$ be an $n$ by $n$ square matrix. If its characteristic equation $\chi_A(t) = 0$ has a repeated root then $A$ may not be diagonalizable, so we need the Jordan Canonical Form. Suppose $\lambda$ is an eigenvalue of $A$, with multiplicity $r$ as a root of $\chi_A(t) = 0$. The the vector $v$ is an eigenvector with eigenvalue $\lambda$ if $A v = \lambda v$ or equivalently

$$(A - \lambda I) v = 0.$$ 

The trouble is that this equation may have fewer then $r$ linearly independent solutions for $v$. So we generalize and say that $v$ is a generalised eigenvector with eigenvalue $\lambda$ if

$$(A - \lambda I)^k v = 0$$

for some positive integer $k$. Now one can prove that there are exactly $r$ linearly independent generalized eigenvectors. Finding the Jordan form is now a matter of sorting these generalized eigenvectors into an appropriate order.

To find the Jordan form carry out the following procedure for each eigenvalue $\lambda$ of $A$. First solve $(A - \lambda I)v = 0$, counting the number $r_1$ of linearly independent solutions. If $r_1 = r$ good, otherwise $r_1 < r$ and we must now solve $(A - \lambda I)^2 v = 0$. There will be $r_2$ linearly independent solutions where $r_2 > r_1$. If $r_2 = r$ good, otherwise solving $(A - \lambda I)^3 v = 0$ gives $r_3 > r_2$ linearly independent solutions, and so on. Eventually one gets $r_1 < r_2 < \cdots < r_{N-1} < r_N = r$. The number $N$ is the size of the largest Jordan block associated to $\lambda$, and $r_1$ is the total number of Jordan blocks associated to $\lambda$. If we define $s_1 = r_1$, $s_2 = r_2 - r_1$, $s_3 = r_3 - r_2$, $\ldots$, $s_N = r_N - r_{N-1}$ then $s_k$ is the number of Jordan blocks of size at least $k$ by $k$ associated to $\lambda$. Finally put $m_1 = s_1 - s_2$, $m_2 = s_2 - s_3$, $\ldots$, $m_{N-1} = s_{N-1} - s_N$ and $m_N = s_N$. Then $m_k$ is the number of $k$ by $k$ Jordan blocks associated to $\lambda$. Once we’ve done this for all eigenvalues then we’ve got the Jordan form!

To find $P$ such that $J = P^{-1} A P$ is the Jordan form then we need to work a bit harder. We do the following for each eigenvalue $\lambda$. First find the Jordan block sizes associated to $\lambda$ by the above process. Put them in decreasing order $N_1 \geq N_2 \geq N_3 \geq \cdots \geq N_k$. Now find a vector $v_{1,1}$ such that $(A - \lambda I)^{N_1} v_{1,1} = 0$ but $(A - \lambda I)^{N_1-1} v_{1,1} \neq 0$. Define $v_{1,2} = (A - \lambda I) v_{1,1}$, $v_{1,3} = (A - \lambda I) v_{1,2}$, and so on until we get $v_{1,N_1}$. We can’t go further as $(A - \lambda I) v_{1,N_1} = 0$. If we only have one block we’re OK, otherwise we can find a vector $v_{2,1}$ such that $(A - \lambda I)^{N_2} v_{2,1} = 0$, $(A - \lambda I)^{N_2-1} v_{2,1} \neq 0$ and (this
is important!) \( \mathbf{v}_{2,1} \) is not linearly dependent on \( \mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,N_1} \). Define \( \mathbf{v}_{2,2} = (A - \lambda I)\mathbf{v}_{2,1} \) etc., until we get to \( \mathbf{v}_{2,N_2} \). If \( k = 2 \) this is the end, if not then choose \( \mathbf{v}_{3,1} \) with \( (A - \lambda I)^{N_3} \mathbf{v}_{3,1} = 0 \), \( (A - \lambda I)^{N_3-1} \mathbf{v}_{3,1} \neq 0 \) and \( \mathbf{v}_{3,1} \) not linearly dependent on \( \mathbf{v}_{1,1}, \ldots, \mathbf{v}_{1,N_1}, \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{2,N_2} \). Keep going! Eventually we get \( r \) linearly independent vectors \( \mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \ldots, \mathbf{v}_{k,N_k} \). Let

\[
P_\lambda = (\mathbf{v}_{k,N_k} \cdots \mathbf{v}_{1,1})
\]

be the \( n \) by \( r \) matrix whose columns are these vectors in reverse order. Once we’ve done this for all eigenvalues \( \lambda \) stick the matrices \( P_\lambda \) together horizontally to get an \( n \) by \( n \) matrix \( P \). Then \( P \) will be non-singular, and \( P^{-1}AP = J \), the Jordan form.

**A worked example**

To illustrate this method, I give a reasonably sized example (6 by 6) which I hope will make things clear, and I hope is safely too big come up on any exam! I have used MAPLE in the computations; only a truly hardy soul would try this one by hand!

Let

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & -1 \\
0 & -8 & 4 & -3 & 1 & -3 \\
-3 & 13 & -8 & 6 & 2 & 9 \\
-2 & 14 & -7 & 4 & 2 & 10 \\
1 & -18 & 11 & -11 & 2 & -6 \\
-1 & 19 & -11 & 10 & -2 & 7
\end{pmatrix}
\]

The characteristic polynomial of this matrix is

\[
\chi_A(t) = t^6 + 3t^5 - 10t^3 - 15t^2 - 9t - 2 = (t + 1)^5(t - 2)
\]

and so its eigenvalues are \(-1\) with multiplicity 5, and \(2\) with multiplicity 1. I’ll deal with \( \lambda = -1 \) first. We first solve \((A + I)v = 0\). The matrix

\[
A + I = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & -1 \\
0 & -7 & 4 & -3 & 1 & -3 \\
-3 & 13 & -7 & 6 & 2 & 9 \\
-2 & 14 & -7 & 5 & 2 & 10 \\
1 & -18 & 11 & -11 & 3 & -6 \\
-1 & 19 & -11 & 10 & -2 & 8
\end{pmatrix}
\]
has REF
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 & 1 & 3/2 \\
0 & 0 & 1 & 0 & 2 & 3/2 \\
0 & 0 & 0 & 1 & 0 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\].

Hence \((A + I)v\) has 2 linearly independent solutions, i.e., \(r_1 = 2\). As \(r_1 < r = 5\) then we must solve \((A + I)^2v = 0\). Now

\[
(A + I)^2 = \begin{pmatrix}
1 & -1 & 0 & 1 & -2 & -3 \\
-2 & -16 & 9 & -11 & 4 & -3 \\
-1 & 37 & -18 & 17 & 2 & 21 \\
1 & 35 & -18 & 19 & -2 & 15 \\
-1 & -53 & 27 & -28 & 2 & -24 \\
2 & 52 & -27 & 29 & -4 & 21 \\
\end{pmatrix}
\]

whose REF is

\[
\begin{pmatrix}
1 & 0 & -1/2 & 3/2 & -2 & -5/2 \\
0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\].

The system \((A + I)^2v\) has \(r_2 = 4\) linearly independent solutions. As \(r_2 < r\), then we now consider \((A + I)^3v\). Now

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -54 & 27 & -27 & 0 & -27 \\
0 & 108 & -54 & 54 & 0 & 54 \\
0 & 108 & -54 & 54 & 0 & 54 \\
0 & -162 & 81 & -81 & 0 & -81 \\
0 & 162 & -81 & 81 & 0 & 81 \\
\end{pmatrix}
\]
and it’s easy to see (!) that the REF of this matrix is
\[
\begin{pmatrix}
0 & 1 & -1/2 & 1/2 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
Hence \((A + I)^3\mathbf{v} = 0\) has \(r_3 = 5\) linearly independent solutions, and as \(r_3 = r\) we conclude this part of the proceedings. We calculate \(s_1 = r_1 = 2\), \(s_2 = r_2 - r_1 = 2\) and \(s_3 = r_3 - r_2 = 1\); also \(m_3 = s_3 = 1\), \(m_2 = s_2 - s_3 = 1\) and \(m_1 = s_1 - s_2 = 0\). Hence, associated to \(\lambda = -1\), there is a 2 by 2 and a 3 by 3 Jordan block. As the other eigenvalue \(\lambda = 2\) has multiplicity 1, then there’s just a 1 by 1 Jordan block associated to \(\lambda = 2\). Hence the Jordan canonical form of \(A\) is
\[
J = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]
Let’s compute the matrix \(P\). We’ve already done most of the work for \(\lambda = -1\). The Jordan blocks have sizes \(N_1 = 3\) and \(N_2 = 2\). We start by finding a vector \(\mathbf{v}_{1,1}\) with \((A + I)^3\mathbf{v}_{1,1} = 0\) but \((A + I)^2\mathbf{v}_{1,1} \neq 0\). Looking at the REFs of these matrices we see that we can choose
\[
\mathbf{v}_{1,1} = (1\ 0\ 0\ 0\ 0\ 0)^t.
\]
Now
\[
\mathbf{v}_{1,2} = (A + I)\mathbf{v}_{1,1} = (1\ 0\ -3\ -2\ 1\ -1)^t
\]
and
\[
\mathbf{v}_{1,3} = (A + I)\mathbf{v}_{1,2} = (1\ -2\ -1\ 1\ -1\ -2)^t.
\]
(As a check one verifies \((A + I)\mathbf{v}_{1,3} = 0\).) The next block is 2 by 2, so one must find \(\mathbf{v}_{2,1}\) with \((A + I)^2\mathbf{v}_{2,1} = 0\), \((A + I)\mathbf{v}_{2,1} \neq 0\), and such that \(\mathbf{v}_{2,1}\) is not linearly dependent on \(\mathbf{v}_{1,1}, \mathbf{v}_{1,2}\) and \(\mathbf{v}_{1,3}\). The vector
\[
\mathbf{v}_{2,1} = (1\ 1\ 2\ 0\ 0\ 0)^t
\]
fits the bill, and

\[
v_{2,2} = (A + I)v_{2,1} = (1 \ 1 -4 \ -2 \ 5 \ -4)^t.
\]

Again one checks that \((A + I)v_{2,2} = 0\). The matrix \(P_{-1}\) is the 6 by 5 matrix with columns \(v_{2,2}, v_{2,1}, v_{1,3}, v_{1,2}\) and \(v_{1,1}\) in that order and so

\[
P_{-1} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & -2 & 0 & 0 \\
-4 & 2 & -1 & -3 & 0 \\
-2 & 0 & 1 & -2 & 0 \\
5 & 0 & -1 & 1 & 0 \\
-4 & 0 & 2 & -1 & 0
\end{pmatrix}.
\]

One must now consider \(\lambda = 2\). As this is a simple root, \(P_2\) is just an eigenvector with eigenvalue 2. One such is

\[
P_2 = (0 \ 1 -2 -2 3 -3)^t
\]

and sticking together \(P_{-1}\) and \(P_2\) gives

\[
P = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & -2 & 0 & 0 & 1 \\
-4 & 2 & -1 & -3 & 0 & -2 \\
-2 & 0 & 1 & -2 & 0 & -2 \\
5 & 0 & -1 & 1 & 0 & 3 \\
-4 & 0 & 2 & -1 & 0 & -3
\end{pmatrix}.
\]

One now checks that \(P^{-1}AP = J\) as required!

RJC 25/1/95