Let $K$ be an algebraically closed field unless otherwise noted.

1. (Harris 6.4) For any point $p \in \mathbb{P}^3$ and any plane $H \subset \mathbb{P}^3$ containing $p$ let $\Sigma_{p,H} \subset \mathbb{G}(1,3)$ be the set of lines in $\mathbb{P}^3$ passing through $p$ and lying in $H$. Show that under the Plucker imbedding $\mathbb{G}(1,3) \to \mathbb{P}^5$ the subvariety $\Sigma_{p,H}$ is mapped to a line, and conversely every line in $\mathbb{P}^5$ lying on $\mathbb{G}(1,3)$ is of the form $\Sigma_{p,H}$ for some $p$ and $H$.

2. (Harris 6.5) For a point $p \in \mathbb{P}^3$ let $\Sigma_p \subset \mathbb{G}(1,3)$ be the set of lines in $\mathbb{P}^3$ passing through $p$. For any plane $H \subset \mathbb{P}^3$ let $\Sigma_H \subset \mathbb{P}^3$ be the set of lines in $\mathbb{P}^3$ contained in $H$. Show that the Plucker embedding carries both $\Sigma_p, \Sigma_H$ into 2–planes in $\mathbb{P}^5$. Show that conversely any 2–plane $\Lambda \simeq \mathbb{P}^2 \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ either equals $\Sigma_p$ for some $p$ or $\Sigma_H$ for some $H$.

3. (Harris 6.6) Let $\ell_1, \ell_2 \subset \mathbb{P}^3$ be skew lines (that is nonintersecting lines). Show that the set $Q \subset \mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ meeting both is the intersection of $\mathbb{G}(1,3)$ with a three–plane $\mathbb{P}^3 \subset \mathbb{P}^5$ and hence is a quadric surface. Deduce that $Q$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Do the same problem for lines $\ell_1, \ell_2 \subset \mathbb{P}^3$ which meet.

4. (Harris 6.20) Let $Q \subset \mathbb{P}^3$ be the zero set of $Z_0Z_3 - Z_1Z_2$. Show that the Fano variety $F_1(Q)$ of all lines contained in $Q$ is a union of two conic curves. Compare this with the parametric description of the Fano variety given in the discussion of the Segre map.

5. (Harris 7.6) Show that if $\phi : X \to \mathbb{P}^n$ is a rational map and $Z \subset X$ is a subvariety such that the restriction $\eta$ of $\phi$ to $Z$ is defined, then the graph $\Gamma_\eta \subset Z \times \mathbb{P}^n$ is contained in the set $\pi^{-1}_1(Z) = \Gamma_\phi \cap (Z \times \mathbb{P}^n) \subset X \times \mathbb{P}^n$ but may not equal it. Thus the proper transform of $Z$ can be properly contained in the total transform of $Z$.

6. (Harris 7.7) Show that the image of a rational normal curve $C \subset \mathbb{P}^n$ under projection from a point $p \in C$ is a rational normal curve in $\mathbb{P}^{n-1}$. Is this true if we project from a point not in $C$?