Problem Set 9 (Last revised 11/18/2016)

Let K be an algebraically closed field unless otherwise noted.

1. (Harris 6.4)For any point $p \in \mathbf{P}^3$ and any plane $H \subset \mathbf{P}^3$ containing p let $\Sigma_{p,H} \subset \mathbf{G}(1,3)$ be the set of lines in $\mathbf{P}^3$ passing through p and lying in H. Show that under the Plucker imbedding $\mathbf{G}(1,3) \to \mathbf{P}^5$ the subvariety $\Sigma_{p,H}$ is mapped to a line, and conversely every line in $\mathbf{P}^5$ lying on $\mathbf{G}(1,3)$ is of the form $\Sigma_{p,H}$ for some p and H.
2. (Harris 6.5) For a point $p \in \mathbf{P}^3$ let $\Sigma_p \subset \mathbf{G}(1,3)$ be the set of lines in $\mathbf{P}^3$ passing through p. For any plane $H \subset \mathbf{P}^3$ let $\Sigma_H \subset \mathbf{P}^3$ be the set of lines in $\mathbf{P}^3$ contained in H. Show that the Plucker embedding carries both Σ_p, Σ_H into 2–planes in $\mathbf{P}^5$. Show that conversely any 2–plane $\Lambda \simeq \mathbf{P}^2 \subset \mathbf{G}(1,3) \subset \mathbf{P}^5$ either equals Σ_p for some p or Σ_H for some H.
3. (Harris 6.6) Let $\ell_1, \ell_2 \subset \mathbf{P}^3$ be skew lines (that is nonintersecting lines). Show that the set $Q \subset \mathbf{G}(1,3)$ of lines in $\mathbf{P}^3$ meeting both is the intersection of $\mathbf{G}(1,3)$ with a three–plane $\mathbf{P}^3 \subset \mathbf{P}^5$ and hence is a quadric surface. Deduce that Q is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Do the same problem for lines $\ell_1, \ell_2 \subset \mathbf{P}^3$ which meet.
4. (Harris 6.20) Let $Q \subset \mathbf{P}^3$ be the zero set of $Z_0Z_3 - Z_1Z_2$. Show that the Fano variety $F_1(Q)$ of all lines contained in Q is a union of two conic curves. Compare this with the parametric description of the Fano variety given in the discussion of the Segre map.
5. (Harris 7.6) Show that if $\phi : X - - \to \mathbf{P}^n$ is a rational map and $Z \subset X$ is a subvariety such that the restriction η of ϕ to Z is defined , then the graph $\Gamma_\eta \subset Z \times \mathbf{P}^n$ is contained in the set $\pi_1^{-1}(Z) = \Gamma_\phi \cap (Z \times \mathbf{P}^n) \subset X \times \mathbf{P}^n$ but may not equal it. Thus the proper transform of Z can be properly contained in the total transform of Z.
6. (Harris 7.7) Show that the image of a rational normal curve $C \subset \mathbf{P}^n$ under projection from a point $p \in C$ is a rational normal curve in $\mathbf{P}^{n-1}$. Is this true if we project from a point not in C?