Problem Set 5 (Last revised 10/27/2016)

Recall that a morphism $\phi : X \to Y$ for projective varieties X, Y is a continuous function such that for each open set U of Y and regular function f on U the composition $f \circ \phi$ is regular on $\phi^{-1}(U)$.

Let K be an algebraically closed field.

1. Verify that if $X \subset \mathbf{P}^m$ is a projective variety and $F_i(x_0, \ldots, x_m), i = 0, \ldots, n$ are homogeneous polynomials of degree d with no common zero on X then the function $\phi([x_0, \ldots, x_m]) = [F_0(x_0, \ldots, x_m), \ldots, F_n(x_0, \ldots, x_m)]$ is a well defined function from X to $\mathbf{P}^n$. By considering the restriction of ϕ to the open subsets of the affine variety $X \cap \{[x_0, \ldots, x_m] | x_j \neq 0\}$ where $F_i(x_0, \ldots, n_m) \neq 0$ show that ϕ is a morphism from X to $\mathbf{P}^n$ (see the middle paragraph on page 21 of Harris).
2. The goal of this problem is to determine the set of morphisms from $X = \mathbf{P}^1$ to $Y = \mathbf{P}^1$. Take homogeneous coordinates $[x_0, x_1]$ on X and $[s, t]$ on Y.
   a) Suppose that r(z) is a rational function of the variable z, that is a member of the fraction field $K(z)$ of the polynomial ring $K[z]$ ($K(z)$ is the localization of $K[z]$ with respect to the multiplicative set of nonzero polynomials). If $r(z) = f(z)/g(z)$ is an expression for the rational function where $f(z), g(z)$ have no common root, define the degree d of $r(z)$ to be the maximum of degrees of $f(z), g(z)$. Show that the rational functions $F = x_1^d f(x_0/x_1), G = x_1^d g(x_0/x_1)$ are homogeneous polynomials in x_0, x_1 of degree d, which factor as the product of d linear polynomials, and have no common zeros on P^1. Let $\phi_r([x_0, x_1]) = [F(x_0, x_1), G(x_0, x_1)]$ be the morphism from $\mathbf{P}^1$ to $\mathbf{P}^1$ defined in problem 1.
   b) Show that the construction of (a) gives a one to one map of the monoid of rational functions under composition to the monoid of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^1$.
   c) Show that every morphism ϕ from $\mathbf{P}^1$ to itself arises from the construction in (a). Hint: on some nonempty open set in X show that $\phi([x_0, x_1])$ is a rational function r of x_1/x_0 and verify that $\phi = \phi_r$.
   d) Use the preceeding to show that the group of automorphisms $Aut(\mathbf{P}^1)$ is isomorphic to the group of degree 1 rational functions under composition and that this is isomorphic to $PGL(2, K)$.
3. (Harris 2.8) Show that the image $Y = \nu_d(X) \subset \mathbf{P}^N$ of a projective variety $X \subset \mathbf{P}^n$ is isomorphic to X via the Veronese map.
4. (Harris 2.9) Use 2.8 to show that any projective variety is isomorphic to the intersection of a Veronese variety and a linear subspace, and hence is isomorphic to an intersection of quadrics.
5. (Harris 2.10) Let $Y = \nu_d(X) \subset \mathbf{P}^N$ be the Veronese image of a projective variety $X \subset \mathbf{P}^n$. What is the relation of the homogeneous coordinate rings? Show that the homogeneous coordinate ring is not an isomorphism invariant of a projective variety,

and construct varieties in a projective space that are isomorphic but not projectively equivalent.

6. (Harris 2.13) Show that the twisted cubic in $\mathbf{P}^3$ is the intersection of the Segre threefold with a three plane in $\mathbf{P}^5$.

7. (Harris 2.15) Show that the image of the diagonal $\Delta \subset \mathbf{P}^n \times \mathbf{P}^n$ under the Segre map is the Veronese variety $\nu_2(\mathbf{P}^n)$ lying in a subspace of $\mathbf{P}^{n^2+2n}$. Conclude that in general the diagonal $\Delta_X \subset X \times X$ is a subvariety of the product variety, and similarly for the diagonal in the n–fold product X^n.

8. (Harris 2.24) Let $X \subset \mathbf{P}^n$ be a projective variety and let $\phi : X \to \underline{\mathrm{P}}^m$ be any regular map. Show that the graph $\Gamma_\phi \subset X \times \mathbf{P}^n \subset \mathbf{P}^n \times \mathbf{P}^m$ is a subvariety.