Mathematics 535 Algebraic Geometry Fall 2016

Problem Set 3 (Last revised 9/21/2016)

1. (Harris 1.11) For $i = 0, 1, 2$ let F_i be the determinant of the 2 by 2 matrix obtained by removing the $3 - i$ column from

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix}$$

Show that the intersection of two distinct quadric surfaces Q_i which are the zero set of F_i is the union of a line and the twisted cubic. More generally, for $\lambda = [\lambda_0, \lambda_1, \lambda_2]$ a point in the projective plane let Q_λ be the zero set of $\lambda_0 F_0 + \lambda_1 F_1 + \lambda_2 F_2$. Show that for μ, ν distinct the intersection of Q_μ, Q_ν is the union of the twisted cubic and a line $L_{\mu,\nu}$

2. (Harris 1.12) Show that any finite set of points on the twisted cubic is in general position.

3. (Harris 1.13) Let p_i be 7 distinct points on a twisted cubic C. Show that the zero locus of all quadratic polynomials which vanish at all points p_i is precisely C.

4. (Harris 1.15) Let p_i be kd+1 distinct points on a rational normal curve C in $\mathbf{P}^d$. Show that any polynomial F of degree k vanishing on the p_i also vanishes on C. Show that exercise 1.5 is sharp.

5. (Harris 1.19) Define the cross ratio of 4 ordered points P_1, P_2, P_3, P_4 in general position in $\mathbf{P}^1$ to be the image of P_4 under the projective transformation sending P_i to E_i for $i = 1, 2, 3$. Show that P_1, P_2, P_3, P_4 are projectively equivalent (as an ordered set) to P'_1, P'_2, P'_3, P'_4 if and only if the cross ratio of the sets agrees. Generalize this to show that n+3 ordered points $P_1, P_2, P_3, \ldots, P_{n+3}$ in general position in $\mathbf{P}^n$ are projectively equivalent to $P'_1, P'_2, P'_3, \ldots, P'_{n+3}$ if and only if the cross ratio of Q_1, Q_2, Q_3, Q_i equals that of Q'_1, Q'_2, Q'_3, Q'_i for $i = 4, \ldots, n+3$ where the points Q_i are the preimage of the points P_i under the map from $\mathbf{P}^1$ given by the unique rational normal curve through the P_i (similarly for P'_i).

6. (Harris 1.21) Let P be a point in the projective plane and let W_P be the hyperplane in $\mathbf{P}^5$ defined by the condition that the quadratic polynomial $\lambda_0 Z_0^2 + \lambda_1 Z_1^2 + \lambda_2 Z_2^2 + \lambda_3 Z_0 Z_1 + \lambda_4 Z_0 Z_2 + \lambda_5 Z_1 Z_2$ vanishes at P. If P_i are 5 points in $\mathbf{P}^2$ with no 4 collinear show that the hyperplanes W_{P_i} intersect in a single point. Show that this means that there is a unique plane conic curve through the 5 points.