Problem Set 11 (Last revised 11/15/2016)

1. (Harris 11.11) Find a homogeneous quadratic polynomial $Q(Z_0, \dots, Z_3)$ and a homogeneous cubic polynomial $P(Z_0, \dots, Z_3)$ whose common zero locus is a rational normal curve $C \subset \mathbf{P}^3$.

2. (Harris 11.15) Let X, Y be irreducible projective varieties, $f : X \to Z$ and $g : Y \to Z$ surjective maps. Show that the dimension of the fiber product satisfies

$$\dim(X \times_Z Y) \geq \dim(X) + \dim(Y) - \dim(Z).$$

   Find an example where strict inequality holds. Show by example that some irreducible components of $X \times_Z Y$ may have strictly smaller dimension.

3. (Harris 11.20) Let $X \subset \mathbf{P}^n$ be an irreducible non-degenerate k-dimensional variety. For $l < n - k$ find the dimension of the closure of the locus of l-planes meeting X in at least two points. Show by example that no analogous formula exists if we replace two by three, even if we require $l \geq 2$.

4. (Harris 11.21) Let $X \subset \mathbf{P}^n$ be an irreducible non-degenerate k-dimensional variety. Show that the plane spanned by $n - k + 1$ general points of X meets X in only finitely many points and use this to compute the dimension of the closure of the locus of l-planes spanned by their intersections with X when $l \leq n - k$.

5. (Harris 11.25) Show that if $X \subset \mathbf{P}^n$ is an irreducible curve then the chordal(or secant) variety $S(X)$ is three-dimensional unless X is contained in a plane.

6. (Harris 11.31) Let $X \subset \mathbf{P}^n$ be an irreducible nondegenerate variety of dimension k. use 11.21 to deduce that for any $l \leq n - k$ the general fiber of the secant plane map

$$s_l : X^{l+1} \to \mathbf{G}(l, n)$$

   is finite.

7. (Harris 11.39) Show that if $m > n$ there does not exist a nonconstant regular map $\phi : \mathbf{P}^m \to \mathbf{P}^n$ by using 11.38.Hint: First try the case n=1 and look at dimensions of inverse images of a hyperplane and a point not on it.

8. (Harris 11.41) Find the dimension of the flag variety $\mathbf{F}(a_1, \dots, a_n)$ in general.