Problem Set 1A (Last revised 9/4/2016)

1. Let $M_2(\mathbf{C})$ be the space of two by two matrices with entries from K considered as $\mathbf{A}^4(\mathbf{C})$ via the 4 matrix entries as coordinates. Let X be the set of matrices B such that $B^2 = B$. Show that X is a reducible algebraic variety. Find the irreducible components of X. Hint: Recall the result proved in class that over an infinite field the similarity class of a square matrix is an irreducible set.

2. Let K be a field, V an n-dimensional vector space over K with basis $e_1, \ldots, e_n$ and let $\wedge V$ be the exterior algebra (the K vector space with basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$, $1 \le k \le n$ and algebra structure satisfying $w \wedge w = 0$.) Let $\wedge^k V$ be the span of vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \le i_1 < \cdots < i_k \le n$. Show that the endomorphism of V represented by a matrix A induces a endomorphism of $\wedge^k V$ and denote the matrix of this endomorphism by $\wedge^k A$.
   a) Show that the matrix entries of $\wedge^k A$ are polynomials in the entries of.A
   b) Discover the coefficients of the characteristic polynomial of A in terms of traces of $\wedge^k A$ by first computing on a convenient Zariski dense set of matrices and using the irreducibility of $A^{n^2}(K)$ to verify your formula holds for all matrices.

3. Let $F_n(x, y), F_{n-1}(x, y)$ be homogeneous polynomials of degrees n and $n - 1$ respectively in $K[x, y]$ for a field K. Show that the zero set of any irreducible polynomial of the form $F(x, y) = F_n(x, y)+F_{n-1}(x, y)$ is a rational variety. Use this to parameterize the zero sets of $y^2 - x^3, y^2 - x^3 - x^2$, the Folium of Descartes $x^3 + y^3 - 3xy$, the 5 leaved rose $(x^2 + y^2)^3 - 5x^4y + 10x^2y^3 - y^5$ (and more generally $r = \sin n\theta$ for odd n).

4. Find a rational parameterization of the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. Hint: compute the intersection of the lemniscate with the family of circles $x^2+y^2 = t(x-y)$.

5. Show that the zero set of $zy^2 - x^2$ is rational. (This affine cubic surface is called Whitney's umbrella). More generally, any projective cubic surface F=0 containing a point P with the gradient of F vanishing at P and such that lines joining P to points on the surface do not all lie wholly in the surface is rational.

Proof:

6. Let R be a Boolean ring, that is $r^2 = r$ for all $r \in R$.
   a) Show that R is a commutative ring of characteristic 2 and that all prime ideals are maximal.
   b) Show that spec(R) is a compact Hausdorff space in which every connected component is a point. (a so called Stone space)
   c) Show that if X is compact Hausdorff and totally disconnected then the ring R of continuous functions from X to the field with two elements (considered as a topological space with the discrete topology) is a Boolean algebra R. Show that X is homeomorphic to spec(R).

d) Show that the constructions above yield an equivalence of categories between Stone spaces with continuous maps, and Boolean algebras with ring homomorphisms. Since a homomorphism $R \to S$ yields by inverse image a map $spec(S) \to spec(R)$ this relation is a duality (introduced by Marshall Stone in 1936).

6. Let $V \subset \mathbf{A}^n(K)$ and $W \subset \mathbf{A}^m(K)$ be Zariski closed subsets. Show that the product set $V \times W = \{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbf{A}^{n+m} | (x_1, \ldots, x_n) \in V, (y_1, \ldots, y_m) \in W\}$ is Zariski closed in $\mathbf{A}^{m+n}$.
   a) Show that although as a set $\mathbf{A}^n(K) \times \mathbf{A}^m(K)$ is the same as $A^{m+n}(K)$ there are subsets of $A^{m+n}(K)$ which are closed in the Zariski topology but not in the product topology.
   b) Show that if V, W as above are irreducible, then $V \times W$ is irreducible. Hint: If the product is a union of closed sets Z_1, Z_2 show that the intersection of vertical sets $V \times \{w\}$ with the closed set Z_i in $\mathbf{A}^{m+n}$ is wholly contained in some Z_i. Show that the set W_i of all $w \in W$ such that $V \times \{w\}$ is contained in Z_i is closed.
   c) Use the preceeding to show that if $W \subset M_n(K)$ is an irreducible set and V is an irreducible set of invertible matrices then the union of the conjugates of elements of W by matrices in V is an irreducible set.
   d) Show that the set of upper triangular matrices of determinant 1 in $M_n(K)$ is an irreducible variety when K is infinite.
   Show every matrix in $M_n(K)$ is similar to an upper triangular matrix when K is algebraically closed. Show that the set $SL(n, K)$ of matrices of determinant 1 with entries from an algebraically closed field K is an irreducible affine variety. (It turns out that an algebraic group is irreducible if and only if it is connected. Since in the representation theory of groups irreducible is used to denote a simple module over the group ring, we generally refer to a connected algebraic group intstead of an irreducible one. So far we have seen that $GL(n, K), SL(n, k)$ are connected alebraic groups.)