Equivalence of various definitions of dimension

There are several definitions of dimension of an algebraic variety. The purpose of this note is to show that they all agree. The basic idea is that dimension should not increase under dominant rational maps, that dimension is a birational invariant, and that linear spaces missing a variety in $\mathbf{P}^n$ can be used to give morphisms to complementary linear spaces.

Proposition 1. *Let X be an irreducible quasiprojective variety in $\mathbf{P}^n$. The following integers are equal:*

a) $k_1 = \sup\{k|$ there exists a dominant rational map $\phi : X - -- > \mathbf{P}^k\}$
b) $k_2 =$ transcendence degree of $K(X)/K$.
c) $k_3 = \min\{k|$ the general (n − k − 1) plane in $\mathbf{P}^n$ misses$X\}$

Proof: The existence of a dominant rational map $\phi : X - -- > \mathbf{P}^{k_1}$ implies that the function field $K(\mathbf{P}^{k_1}) = K(t_1, \ldots, t_{k_1})$ is a subfield of $K(X)$, so that $k_2 \geq k_1$. If $t_1, \ldots, t_{k_2}$ is a transcendence base for $K(X)$ over K the inclusion $K(t_1, \ldots, t_{k_2}) \subset K(X)$ corresponds to a dominant rational map of X to $\mathbf{P}^{k_2}$, so that $k_1 \geq k_2$. Thus $k_1 = k_2$.

Let $\Lambda \subset \mathbf{P}^n$ be an $(n - k_3 - 1)$ dimensional linear space missing X. Let π_Λ be the projection from Λ to a complementary linear space of dimension k_3. The projection is a finite to one morphism. Further, π_Λ is dominant, since a general plane generated by Λ and general points of $\mathbf{P}^{k_3}$ must hit X, and thus the general point of $\mathbf{P}^{k_3}$ is in $\pi_\Lambda(X)$. Then the function field inclusion corresponding to this dominant map shows that the transcendence degree of $K(X)$ is k_3.

We define the dimension of an irreducible quasiprojective variety $X \subset \mathbf{P}^n$ to be the integer defined in 3 ways in the preceding proposition. Note that the transcendence degree definition shows that the dimension is a birational invariant, which is independent of the ambient space $\mathbf{P}^n$ containing X. For quasiprojective varieties $X \subset \mathbf{P}^n$ the integers k_1, k_3 defined above clearly equal the maximal dimension of the irreducible components of X so for any quasiprojective X which is a union of irreducible components X_i we define $\dim X = \max\{\dim X_i\}$. Note that $\dim X = k$ if and only if there is a finite to one dominant rational map of X to $\mathbf{P}^k$, since such a map preserves transcendence degree of function fields. Thus dimension 0 varieties are finite sets of points.

Proposition 2. *Let $Y \subset X$ be a closed irreducible subset of the irreducible quasiprojective variety X. Then $\dim X \geq \dim Y$, with equality if and only if $Y = X$.*

Proof: When $Y \subset X \subset \mathbf{P}^n$ any plane missing X misses Y so the previous proposition shows that $\dim Y \leq \dim X$.

Now suppose that $\dim Y = \dim X$. The proof that $X = Y$ must use both that Y is closed and X is irreducible, since taking Y as the complement of a point in X or taking Y to be one of the many components if X is reducible shows that neither of these conditions can be dropped.

Let F vanish on Y but not on X. Since X is irreducible F does not vanish on any nonempty open subset of X. Given a point in a quasiprojective variety there is an open neighborhood of the point which is isomorphic to an affine variety. Choose a point $y_0 \in Y \subset X$ and an affine open set $U_X \subset X$ containing y_0. Then $Y \cap U_X$ is closed in the affine variety U_X and open dense in Y so that we have $\dim Y \cap U_X = \dim Y, \dim U_X = \dim X$. If we show that $Y \cap U_X = U_X$, then since X is irreducible Y is dense in X, and since Y is closed in X, Y=X.

So it is enough to prove the theorem assuming that $Y \subset X \subset \mathbf{A}^n$ are irreducible affine varieties and $\dim Y = \dim X = m$. If $Y \neq X$ there is a polynomial $F(x_1, ..., x_n)$ vanishing on Y but not on X, since Y is closed in X. Choose elements $t_1, \ldots, t_m$ of $K[x_1, \ldots, x_n]$ which are algebraically independent as rational functions on Y. Consider $t_1, \ldots, t_m, F$ as rational functions on X. Since $\dim X = \dim Y = m$ any nonzero rational function on X, for example F, satisfies an irreducible nonzero polynomial equation with coefficients from $K(t_1, \ldots, t_m)$. Clearing denominators we have that $F^r a_r(t_1, \ldots, t_m) + \cdots + a_0(t_1, \ldots, t_m) = 0 \in K(X)$ where a_i are polynomials in $t_1, \ldots, , t_m$ and a_0 is not the zero polynomial. Since F vanishes on Y, this implies that $t_1, \ldots, t_m$ are algebraically dependent in $K(Y)$ contradicting the choice of t_i. Thus $X = Y$.

We now wish to study the intersection of an irreducible quasiprojective variety X with a hypersurface Z. It may be that $Z \cap X$ is empty or all of X. If not, we wish to show that the dimension of each irreducible component has been reduced by 1 from the dimension of X mimicking the behavior of linear spaces intersected with hyperplanes. Note in particular that the statement 11.6 in Harris is not quite correct - it is already false for an affine line X in $\mathbf{A}^2$ and the line Z at infinity in $\mathbf{P}^2$.

Proposition 3. *Let $X \subset \mathbf{P}^n$ be an irreducible quasiprojective variety and $Z \subset \mathbf{P}^n$ a hypersurface not containing X, then the dimension of each irreducible component of $Z \cap X$ is* $\dim X - 1$.

Proof: We will use the transcendence degree definition of dimension. Since the dimension of X is the same as the dimension of the closure $\bar{X}$, and since the closure of an irreducible component of $X \cap Z$ is an irreducible component of $\bar{X} \cap Z$, it will be enough to treat projective varieties X. Finally, by intersecting with the standard affine subsets of projective space, we need only prove the result for affine varieties intersected with hypersurfaces in affine space $\mathbf{A}^n$. Let f be a regular function on X, which is not 0 or constant on X. By the previous proposition the zero set of f on X has dimension strictly less than $r = \dim(X)$, since any proper closed subvariety $Y \subset X$ has $\dim(Y) < \dim(X)$. It only remains to show that the dimension of any irreducible component of the zero set of f is at least $\dim(X) - 1$. Observe that given a (possibly reducible) variety X_i we can find non constant linear functions that do not vanish identically on any component of X_i (choose a point in each component, and take a hyperplane missing these points). Let X_1 be the zero set of f on X, and inductively apply the above to produce $f_2, f_3, \ldots, f_r$ of the same total degree as f (take powers of the linear functions) and $X_2 \supset X_3 \supset \cdots \supset X_r$ such that f_j is nonzero on each component of X_j, and X_{j+1} is its zero set in X_j. We will show that $f_2, \ldots f_r$ are algebraically independent on X_1, which will prove that every irreducible component of the zero set of a nonconstant regular function on X has dimension $\dim(X) - 1$.

Let $P \in K[T_2, \ldots, T_r]$. We want to show that $R = P(f_2, \ldots, f_r)$ does not vanish identically on any component of X_1. For this it is enough to show that if $Q \in K[X], R \cdot Q = 0$ on X_1, then $Q = 0$ on X_1, for if R vanishes on an irreducible component take Q to be any nonzero element of $K[X]$ vanishing on all other components. By the nullstellensatz the first vanishing condition may be replaced by $Z(f) \subset Z(R \cdot Q)$ which is equivalent to f divides a power of $R \cdot Q$, and the second is equivalent to f divides a power of Q. We can reduce to a purely algebraic statement if we observe that projection from a point in projective space not contained in a variety Z exhibits the homogeneous coordinate ring of the projection as a subring of the homogeneous coordinate ring of Z, and the latter ring is a finite module over the former (this is easy to see for the projection from $[0, 0, \ldots, 0, 1]$ by observing that since this point is not in Z, there is a homogeneous polynomial in $I(Z)$ which is of the form $Z_n^d + \ldots$, and thus the homogeneous coordinate ring of Z is generated by $1, Z_n, \ldots, Z_n^{d-1}$ over the homogeneous coordinate ring of the projection). Thus projection from a linear space is finite in the sense that the homogeneous coordinate ring of the variety is a finite module over the homogeneous coordinate ring of the projection, and hence the same is true for the coordinate rings of affine varieties which are dense in a projective variety. Further, since a projective variety is isomorphic to its Veronese image, if $f_0, \ldots f_s$ are linearly independent forms of the same degree which do not vanish simultaneously on a variety Z the map $[z] \mapsto [f_0(z), \ldots, f_s(z)]$ is a regular map which is given by projection on the Veronese image, and thus is finite in the sense above, and preserves dimension. Hence if we projectivize our construction of $X_1, \ldots, X_r$ earlier we see that (since $\dim(X) = r, \dim(X_i) \leq r - i$) $K[X]$ is a finitely generated module over $K[T_1, \ldots T_r]$ for some algebraically independent T_i. We are thus reduced to the following Lemma.

Lemma. *Let $A \supset K[T_1, \ldots, T_r]$ be a finitely generated $K[T_1, \ldots, T_r]$ module which is a domain, $x = T_1, y = P(T_2, \ldots, T_r) \neq 0, u \in A$. If x divides a power of yu in A, the x divides a power of u.*

Proof: Note that by replacing y, u by powers, it is enough to show that if $x, z \in K[T_1, \ldots, T_r]$ are relatively prime, then x divides zv in A implies that x divides a power of v. Let $zv = xw$, and let $F(T) = T^l + b_1 T^{l-1} + \ldots + b_l$ be the minimal polynomial of $w \in A$ considered in the fraction field. Since A is finite over $K[t_1, \ldots, T_r]$ there exist monic polynomials with coefficients in $K[T_1, \ldots, T_r]$ which w satisfies, and since A is a domain it is easy to see that in the minimal polynomial $b_i \in K[T_1, \ldots, T_r]$. The minimal polynomial of $v = xw/z$ is then

$$(x/z)^l F((z/x)T) = T^l + (xb_1/z)T^{l-1} = \ldots + (x^l b_l/z^l)$$

which has coefficients in $K[T_1, \ldots, T_r]$ since v satisfies a monic polynomial with coefficients in this ring (by finiteness). Thus z^i divides b_i, since z, x are relatively prime. Hence from the minimal polynomial we have x divides v^l as desired.

Corollary.

1. *A quasiprojective variety X contains closed subvarieties of all possible dimensions $k \leq \dim X$.*
2. *Every closed subvariety X of $\mathbf{P}^n$ (respectively $\mathbf{A}^n$) of pure codimension 1 is a hypersurface and $I(X)$ is a principal ideal.*

3. $\dim X$ *is the largest integer* k *such that there is a chain of distinct nonempty closed irreducible subsets* $Y_0 \subset \cdots \subset Y_k \subset X$.
4. *A closed irreducible codimension 1 subvariety* W *of an irreducible quasiprojective variety* X *is an irreducible component of the intersection of* X *with a hypersurface* Z.

Proof: By the previous proposition any nonempty quasiprojective variety contains a closed subvariety of dimension 1 less, continuing this gives closed subvarieties of all dimensions $k \leq \dim X$, establishing 1. Any chain of distinct nonempty closed irreducible subsets $Y_0 \subset \cdots \subset Y_k \subset X$ has $k \leq \dim X$ since the dimension is decreased in a proper containment of irreducible closed subvarieties by Proposition 1. By 1. there are chains with $k = \dim X$, establishing 3. Suppose that $X \subset \mathbf{P}^n$ is a projective variety of pure codimension 1. Let X_i be an irreducible component of X and let F be a nonzero homogeneous polynomial vanishing on X_i. Then $X_i \subset Z(F)$ and $\dim X_i = n-1 = \dim Z(F)$ so that X_i is an irreducible component of $Z(F)$. Hence there is an irreducible polynomial F_i so that $X_i = Z(F_i), I(X_i) = (F_i)$ and thus $X = \cup X_i = Z(F_1 F_2 \cdots F_m)$ and $I(X) = (F_1 F_2 \cdots F_m)$. Similarly, if W is closed irreducible of codimension 1 in an irreducible variety X there is a polynomial F which vanishes on W but not on X so that all components of $X \cap Z(F)$ have codimension 1 in X, and W is contained in some component of this intersection. By Propositions 2 and 3 the variety W equals the component it is contained in since the dimensions are the same, establishing 4.

We can now show the equivalence of the several different definitions of dimension. Let X be a quasiprojective variety contained in $\mathbf{P}^n$. We define the following integers:

Krull dimension of X = maximum of integers k so that there exists a chain of distinct nonempty irreducible closed subvarieties $Y_0 \subset \cdots \subset Y_k \subset X$.

Projection dimension of X = maximum integer k so that there is a dominant rational map from X to $\mathbf{P}^k$.

Section dimension of X = minimum k such that the general $(n-k-1)$ plane does not intersect X

Variant of Section dimension of X = maximum k such that the general $(n-k)$ plane intersects X in a finite set of points.

Transcendence dimension of X= maximum transcendence degree of $K(Y)/K$ for Y irreducible in X.

Theorem. *The integers defined above for a quasiprojective variety* X *are all equal.*

Proof: Proposition 1 establishes that the transcendence dimension, section dimension and projection dimension agree. The Corollary to Proposition 3 shows that the Krull dimension agrees with these. It only remains to observe that when $X \subset P^n$ has dimension k the general $n-i$ plane will intersect X in a $\dim X - i$ dimensional space since the plane is the intersection of i hypersurfaces. Thus the general $n - \dim X$ plane hits X in a finite set, establishing the variant of the section dimension statement.

Corollary. *Suppose that* X *and* Y *are affine varieties in* $\mathbf{A}^n$. *Every irreducible component of* $X \cap Y$ *has dimension at least* $\dim X + \dim Y - n$. *If* V *and* W *are projective varieties*

in $\mathbf{P}^n$ *then every component of* $V \cap W$ *has dimension at least* $\dim V + \dim W - n$*, and* $V \cap W$ *is not empty when* $\dim V + \dim W \geq n$

Proof: By considering irreducible components it we can assume that X, Y are irreducible. Consider $X \times Y$ as a subvariety of $\mathbf{A}^n \times \mathbf{A}^n = \mathbf{A}^{2n}$. The function field of $X \times Y$ is the tensor product $K(X) \otimes K(Y)$ which has transcendence degree $\dim X + \dim Y$. Thus $\dim X \times Y = \dim X + \dim Y$. Consider the hyperplanes $x_i = y_i$ in $\mathbf{A}^n \times \mathbf{A}^n$. By repeated use of Proposition 3 an irreducible component of the intersection of $X \times Y$ with these hyperplanes has dimension at least $\dim X + \dim Y - n$, and since the hyperplanes define the diagonal in $\mathbf{A}^n \times \mathbf{A}^n$ the intersection with the hyperplanes is isomorphic to $X \cap Y$.

For the projective statement, cover $\mathbf{P}^n$ by affine spaces and use the previous paragraph. The final statement follows by considering the affine cones V_{aff}, W_{aff} over V, W in $\mathbf{A}^{n+1}$. These are the affine varieties defined in $\mathbf{A}^{n+1}$ by the homogeneous polynomials in $x_0, ..., x_n$ giving the varieties. Given any chain of distinct nonempty closed irreducibles $Y_0 \subset \cdots \subset Y_k \subset V$ the affine cones give a chain that can be lengthened by placing the point $(0, \ldots, 0)$ at the first. Thus $\dim(V_{aff}) \geq \dim + 1$. Since the vertex $(0, \ldots, 0)$ is in $V_{aff} \cap W_{aff}$ any irreducible component of the intersection containing $(0, \ldots, 0)$ is of dimension at least $\dim V + \dim W - n + 1$ by the previous paragraph. When $\dim V + \dim W \geq n$ there is a component of $V_{aff} \cap W_{aff}$ that contains $(0, \ldots, 0)$ and has dimension at least 1, so it contains nonzero points $(p_0, ..., p_n)$. The image $[p_1, \ldots, p_n] \in \mathbf{P}^n$ is in V and W.