Mathematics 535 Algebraic Geometry Fall 2008

Problem Set 5B (Last revised 11/4/2008)

6.4 For any point $p \in \mathbf{P}^3$ and any plane $H \subset \mathbf{P}^3$ containing p let $\Sigma_{p,H} \subset \mathbf{G}(1,3)$ be the set of lines in $\mathbf{P}^3$ passing through p and lying in H. Show that under the Plucker imbedding $\mathbf{G}(1,3) \to \mathbf{P}^5$ the subvariety $\Sigma_{p,H}$ is mapped to a line, and conversely every line in $\mathbf{P}^5$ lying on $\mathbf{G}(1,3)$ is of the form $\Sigma_{p,H}$ for some p and H.

6.5 For a point $p \in \mathbf{P}^3$ let $\Sigma_p \subset \mathbf{G}(1,3)$ be the set of lines in $\mathbf{P}^3$ passing through p. For any plane $H \subset \mathbf{P}^3$ let $\sigma_H \subset \mathbf{P}^3$ be the set of lines in $\mathbf{P}^3$ contained in H. Show that the Plucker embedding carries both Σ_p, Σ_H into 2–planes in $\mathbf{P}^5$. Show that conversely any 2–plane $\Lambda \simeq \mathbf{P}^2 \subset \mathbf{G}(1,3) \subset \mathbf{P}^5$ either equals Σ_p for some p or Σ_H for some H.

6.6 Let $\ell_1, \ell_2 \subset \mathbf{P}^3$ be skew lines (that is nonintersecting lines). Show that the set $Q \subset \mathbf{G}(1,3)$ of lines in $\mathbf{P}^3$ meeting both is the intersection of $\mathbf{G}(1,3)$ with a three–plane $\mathbf{P}^3 \subset \mathbf{P}^5$ and hence is a quadric surface. Deduce that Q is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Do the same problem for lines $\ell_1, \ell_2 \subset \mathbf{P}^3$ which meet.

6.8 Let $\Sigma_{1,k} \simeq \mathbf{P}^1 \times \mathbf{P}^k \subset \mathbf{P}^{2k+1}$ be the Segre variety, and let Λ_p be the fiber over $p \in \mathbf{P}^1$. Show that Λ_p is a k-plane in $\mathbf{P}^{2k+1}$ and that the assignment $p \mapsto \Lambda_p$ gives a regular map of $\mathbf{P}^1$ to the Grassmannian $\mathbf{G}(k, 2k+1)$ with image a rational normal curve lying in a (k+1)-plane in $\mathbf{P}(\bigwedge^{k+1} K^{2k+2})$.

6.11 Let $\Lambda \subset \mathbf{P}^n$ be a k-plane and let $I(\Lambda)_d$ be the graded piece of degree d in the homogeneous ideal of all homogeneous polynomials vanishing on Λ. Show that the codimension of $I(\Lambda)_d$ in the vector space S_d of all homogeneous polynomials of degree d is $\binom{k+d}{d}$. Show that the map

$$\nu_d : G(K,n) \to G(\binom{k+d}{d}, \binom{n+d}{d})$$

obtained by associating to a k-plane the subspace of linear functionals on S_d which vanish on $I(\Lambda)_d$ is regular.

6.20 Let $Q \subset \mathbf{P}^3$ be the zero set of $Z_0Z_3 - Z_1Z_2$. Show that the Fano variety $F_1(Q)$ of all lines contained in Q is a union of two conic curves. Compare this with the parametric description of the Fano variety given by using the Segre map.