Recall that a morphism \( \phi : X \to Y \) for projective varieties \( X, Y \) is a continuous function such that for each open set \( U \) of \( Y \) and regular function \( f \) on \( U \) the composition \( f \circ \phi \) is regular on \( \phi^{-1}(U) \).

Let \( K \) be an algebraically closed field.

1. Verify that if \( X \subset \mathbb{P}^n \) is a projective variety and \( F_i(x_0, \ldots, x_m), i = 0, \ldots, n \) are homogeneous polynomials with no common zero on \( X \) then the function \( \phi([x_0, \ldots, x_m]) = [F_0(x_0, \ldots, x_m), \ldots, F_n(x_0, \ldots, x_m)] \) is a well defined function from \( X \) to \( \mathbb{P}^n \). By considering the restriction of \( \phi \) to the open subsets of the affine variety \( X \cap \{(x_0, \ldots, x_m)|x_j \neq 0\} \) where \( F_i(x_0, \ldots, x_m) \neq 0 \) show that \( \phi \) is a morphism from \( X \) to \( \mathbb{P}^n \) (see the middle paragraph on page 21 of Harris).

2. The goal of this problem is to determine the set of morphisms from \( X = \mathbb{P}^1 \) to \( Y = \mathbb{P}^1 \). Take homogeneous coordinates \([x_0, x_1]\) on \( X \) and \([s, t]\) on \( Y \).
   a) Suppose that \( r(z) \) is a rational function of the variable \( z \), that is a member of the fraction field \( K(z) \) of the polynomial ring \( K[z] \) (\( K(z) \) is the localization of \( K[z] \) with respect to the multiplicative set of nonzero polynomials). If \( r(z) = f(z)/g(z) \) is an expression for the rational function where \( f(z), g(z) \) have no common root, define the degree \( d \) of \( r(z) \) to be the maximum of degrees of \( f(z), g(z) \). Show that the rational functions \( F = x_0^d f(x_1/x_0), G = x_0^d g(x_1/x_0) \) are homogeneous polynomials in \( x_0, x_1 \) of degree \( d \), which factor as the product of \( d \) linear polynomials, and have no common zeros on \( \mathbb{P}^1 \). Let \( \phi_r([x_0, x_1]) = [F(x_0, x_1), G(x_0, x_1)] \) be the morphism from \( \mathbb{P}^1 \) to \( \mathbb{P}^1 \) defined in problem 1.
   b) Show that the construction of (a) gives a one to one map of the monoid of rational functions under composition to the monoid of morphisms from \( \mathbb{P}^1 \) to \( \mathbb{P}^1 \).
   c) Show that every morphism \( \phi \) from \( \mathbb{P}^1 \) to itself arises from the construction in (a). Hint: on some nonempty open set in \( X \) show that \( \phi([x_0, x_1]) \) is a rational function \( r \) of \( x_1/x_0 \) and verify that \( \phi = \phi_r \).
   d) Use the preceeding to show that the group of automorphisms \( Aut(\mathbb{P}^1) \) is isomorphic to the group of degree 1 rational functions under composition and that this is isomorphic to \( PGL(2, K) \).

1.27 Show that the images of the maps \( \mu, \nu : \mathbb{P}^1 \to \mathbb{P}^2 \) given by \( \mu[x_0, x_1] = [x_0^3, x_0 x_1^2, x_1^3] \) and \( \nu[x_0, x_1] = [x_0^3, x_0 x_1^2 - x_0^3, x_1^3 - x_0^2 x_1] \) are algebraic varieties.

1.28 Let \( \nu : \mathbb{P}^1 \to \mathbb{P}^2 \) be given by three homogeneous cubic polynomials. Show that if the polynomials have no common zero, then the image is a hypersurface which is the zero set of a cubic polynomial.

1.29 Let \( \nu_{\alpha, \beta} : \mathbb{P}^1 \to \mathbb{P}^2 \) be given by \( \nu_{\alpha, \beta}([x_0, x_1]) = [x_0^4 - \beta x_0^3 x_1, x_0^3 x_1 - \beta x_0^2 x_1^2, \alpha x_0^2 x_1^3 - x_0 x_1^3, \alpha x_0 x_1^3 - x_1^4] \). Show that the image of this map is a projective variety which is the zero locus of one quadratic and two cubic polynomials.