Problem Set 1A (Last revised 9/5/2008)

1. Let $F_n(x, y), F_{n-1}(x, y)$ be homogeneous polynomials of degrees n and $n-1$ respectively in $K[x, y]$ for a field K. Show that the zero set of any irreducible polynomial of the form $F(x, y) = F_n(x, y) + F_{n-1}(x, y)$ is a rational variety. Use this to parameterize the zero sets of $y^2 - x^3, y^2 - x^3 - x^2$, the Folium of Descartes $x^3 + y^3 - 3xy$, the 5 leaved rose $(x^2+y^2)^3 - 5x^4y + 10x^2y^3 - y^5$ (and more generally $r = \sin n\theta$ for odd n).

Proof:

2. Find a rational parameterization of the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. Hint: compute the intersection of the lemniscate with the family of circles $x^2+y^2 = t(x-y)$.

Proof:

3. Show that the zero set of $zy^2 - x^2$ is rational. (This affine cubic surface is called Whitney's umbrella). More generally, any projective cubic surface F=0 containing a point P with the gradient of F vanishing at P and such that lines joining P to points on the surface do not all lie wholly in the surface is rational.

Proof:

4. Find the tangent space to the Whitney umbrella at a point (a, b, c). What are the singular points of this variety?

Proof: