8. What Is Algebraic Geometry?

R. Hartshorne, Algebraic Geometry, GTM 52, Chapter I.8

Now that we have met some algebraic varieties, and have encountered some of the main concepts about them, it is appropriate to ask, what is this subject all about? What are the important problems in the field, and where is it going? To define algebraic geometry, we could say that it is the study of the solutions of systems of polynomial equations in an affine or projective n-space. In other words, it is the study of algebraic varieties.

In any branch of mathematics, there are usually guiding problems, which are so difficult that one never expects to solve them completely, yet which provide stimulus for a great amount of work, and which serve as yardsticks for measuring progress in the field. In algebraic geometry such a problem is the classification problem. In its strongest form, the problem is to classify all algebraic varieties up to isomorphism. We can divide the problem into parts. The first part is to classify varieties up to birational equivalence. As we have seen, this is equivalent to the question of classifying function fields (finitely generated extension fields) over k up to isomorphism. The second part is to identify a good subset of a birational equivalence class, such the nonsingular projective varieties, and classify them up to isomorphism. The third part is to study how far an arbitrary variety is from one of the good ones considered above. In particular, we want to know (a) how much do you have to add to a nonprojective variety to get a projective variety, and (b) what is the structure of singularities, and how can they be resolved to give a nonsingular variety?

Typically, the answer to any classification problem in algebraic geometry consists of a discrete part and a continuous part. So we can rephrase the problem as follows: define numerical invariants and continuous invariants of algebraic varieties, which allow one to distinguish among nonisomorphic varieties. Another special feature of the classification problem is that often when there is a continuous family of nonisomorphic objects, the parameter space can itself be given a structure of algebraic variety. This is a very powerful method, because then all the techniques of the subject can be applied to the study of the parameter space as well as to the original varieties.

Let us illustrate these ideas by describing what is known about the classification of algebraic curves (over a fixed algebraically closed field k). First, the birational classification. There is an invariant called the genus of a curve, which is a birational invariant, and which takes on all nonnegative values $g \geq 0$. For $g = 0$ there is exactly one birational equivalence class, namely, that of the rational curves (i.e., those curves which are birationally equivalent to $\mathbf{P^1}$). For each $g > 0$ there is a continuous family of birational equivalence classes, which can be parametrized by an irreducible algebraic variety M_g, called the variety of moduli of curves of genus g, which has dimension 1 if $g = 1$, and dimension $3g-3$ if $g > 1$. Curves with $g = 1$ are called elliptic curves. Thus for curves, the birational classification question 1 is answered by giving the genus, which is a discrete invariant, and a point on the variety of moduli, which is a continuous invariant. See Chapter IV for more details.

The second question for curves, namely, to describe all nonsingular projective curves in a given birational equivalence class, has a simple answer, as we have seen, since there is exactly one.

For the third question, we know that any curve can be completed to a projective curve by adding a finite number of points, so there is not much more to say there. As for the classification of singularities of curves, see (V, 3.9.4).

While we are discussing the classification problem, I would like to describe another special case where a satisfactory answer is known, namely, the classification of nonsingular projective surfaces within a given birational equivalence class. In this case one knows that (1) every birational equivalence class of surfaces has a nonsingular projective surface in it, (2) the set of nonsingular projective surfaces with a given function field K/k is a partially ordered set under the relation given by the existence of a birational morphism, (3) any birational morphism $f : X \rightarrow Y$ can be factored into a finite number of steps, each of which is a blowing-up of a point, and (4) unless K is rational (i.e., equal to K or ruled (i.e., K is the function field of a product $\mathbf{P^1} \times \mathbf{C}$, where C is a curve), there is a unique minimal element of this partially ordered set, which is called the minimal model of the function field K (In the rational and ruled cases, there are infinitely many minimal elements, and their structure is also well-known.) The theory of minimal models is is a very beautiful branch of the theory of surfaces. The results were known to the Italians, but were first proved in all characteristics by Zariski [5], [61. See Chapter V for more details.

From these remarks it should be clear that the classification problem is a very fruitful problem to keep in mind while studying algebraic geometry. This leads us to the next question: how does one go about defining invariants of an algebraic variety? So far, we have defined the dimension, and for projective varieties we have defined the Hilbert polynomial, and hence the degree and the arithmetic genus p_a. Of course the dimension is a birational invariant. But the degree and the Hilbert polynomial depend on the embedding in projective space, so they are not even invariants under isomorphism of varieties. Now it happens that the arithmetic genus is an invariant under isomorphism (111, Ex. 5.3), and is even a birational invariant in most cases(curves, surfaces, nonsingular varieties in characteristics see (V, 5.6.1)), but this is not at all apparent from our definition. To go further, we must study the intrinsic geometry on a variety, which we have not done at all yet. So, for example, we will study divisors on a variety X. A divisor is an element of the free abelian group generated by the subvarieties of codimension one. We will define linear equivalence of divisors, and then we can form the group of divisors modulo linear equivalence, called the Picard group of X. This is an intrinsic invariant of X. Another very important notion is that of a differential form on a variety X. Using differential forms, one can give an intrinsic definition of the tangent bundle and cotangent bundle on an algebraic variety. Then one can carry over many constructions from differential geometry to define numerical invariants. For example, one can define the genus of a curve as the dimension of the vector space of global differential forms on the nonsingular projective model. From this definition it is clear that it is a birational invariant. See (11, 6,7,8). Perhaps the most important modern technique for defining numerical invariants is by cohomology. There are many cohomology theories, but we will be principally concerned in this book with the cohomology of coherent sheaves, which was introduced by Serre [3]. Cohomology is an extremely powerful and versatile tool. Not only can it be used to define numerical invariants (for example, the genus of a curve X can be defined as dim HI(X,(9x)), but it can be used to prove many important results which do not apparently have any connection with cohomology, such as "Zariski's main theorem," which has to do with the structure of birational transformations. To set up a cohomology theory requires a lot of work, but I believe it is well worth the effort. We will devote a whole chapter to cohomology later in the book (Chapter 111). Cohomology is also a useful vehicle for understanding and expressing important results such as the Riemann-Roch theorem 1. This was known classically for curves and surfaces, but it was by using cohomology that Hirzebruch [1] and Grothendieck (see Borel and Serre [1]) were able to clarify and generalize it to varieties of any dimension (Appendix A). Now that we have seen a little bit of what algebraic geometry is about, we should discuss the degree of generality in which to develop the foundations of the subject. In this chapter we have worked over an algebraically closed field, because that is the simplest case. But there are good reasons for .owing fields which are not algebraically closed. One reason is that the local ring of a subvariety on a variety has a residue field which is not algebraically closed (Ex. 3.13), and at times it is desirable to give a unified treatment of properties which hold along a subvariety and properties which Id at a point. Another strong reason for allowing non-algebraically close I fields is that many problems in algebraic geometry are motivated by number theory, and in number theory one is primarily concerned with solutions of equations over finite fields or number fields. For example, Fermat's problem is equivalent to the question, does the curve $x^n + y^n = z^n$ in $\mathbf{P^2}$ for $n \geq 3$ have any points rational over Q (i.e., points whose coordinates are in Q), with $x, y, z \neq 0$.

The need to work over arbitrary ground fields was recognized by Zariski and Weil. In fact, perhaps one of the principal contributions of Weil's "Foundations" [1] was to provide a systematic framework for studying varieties over arbitrary fields, and the various phenomena which occur with change of ground field. Nagata [2] went further by developing the foundations of algebraic geometry over Dedekind domains. Another direction in which we need to expand our foundations is to define some kind of abstract variety which does not a priori have an embedding in an affine or projective space. This is especially necessary in problems such as the construction of a variety of moduli, because there one may be able to make the construction locally, without knowing anything about a global embedding. In 6 we gave a definition of an abstract curve. In higher dimensions that method does not work, because there is no unique not singular model of a given function field. However, we can define an abstract variety by starting from the observation that any variety has an open covering by affine varieties. Thus one can define an abstract variety as a topological space X, with an open cover U_i, plus for each U_i a structure of affine variety, such that on each intersection $U_i \cap U_j$ the induced variety structures are isomorphic. It turns out that this generalization of the notion of variety is not illusory, because in dimension ≥ 2 there are abstract varieties which are not isomorphic to any quasi-projective variety (11, 4.10.2). There is a third direction in which it is useful to expand our notion of algebraic variety. In this chapter we have defined a variety as an irreducible algebraic set in affine or projective space. But it is often convenient to allow reducible algebraic sets, or even algebraic sets with multiple components. For example, this is suggested by what we have seen of intersection theory in section 7, since the intersection of two varieties may be reducible, and the sum of the ideals of the two varieties may not be the ideal of the intersection. So one might be tempted to define a "generalized projective variety" in $\mathbf{P^n}$ to be an ordered pair $< V, I >$, where V is an algebraic set in $\mathbf{P^n}$, and $I \subset S = k[x_0, \ldots, x_n]$ is any ideal such that $V = Z(I)$. This is not in fact what we will do, but it gives the general idea. All three generalizations of the notion of variety suggested above are contained in Grothendieck's definition of a scheme. He starts from the observation that an affine variety corresponds to a finitely generated integral domain over a field (3.8). But why restrict one's attention to such a special class of rings? So for any commutative ring A, he defines a topological space Spec A, and a sheaf of rings on Spec A, which generalizes the ring of regular functions on an affine variety, and he calls this an affine scheme. An arbitrary scheme is then defined by glueing together affine schemes, thus generalizing the notion of abstract variety we suggested above. One caution about working in extreme generality. There are many advantages to developing a theory in the most general context possible. In the case of algebraic geometry there is no doubt that the introduction of schemes has revolutionized the subject and has made possible tremendous advances. On the other hand, the person who works with schemes has to carry a considerable load of technical baggage with him: sheaves, abelian categories, cohomology, spectral sequences, and so forth. Another more serious difficulty is that some things which are always true for varieties may no longer be true. For example, an affine scheme need not have finite dimension, even if its ring is noetherian. So our intuition must be supported by a good knowledge of commutative algebra. In this book we will develop the foundations of algebraic geometry using the language of schemes, starting with the next chapter.