Chapter 4

The 27 lines on a cubic surface

Whole books have been written on the 27 lines on a general cubic surface, for example [Hen], [Seg]. Still this configuration retains its charm, and in a sense, the rest of the book is devoted to some of the geometry which canonically arises from this beautiful configuration. In fact, a lot of the geometry, including all of Chapter 6, is essentially new, giving a complement to well-known results.

Virtually all the geometry is closely tied up with the ***automorphism group*** of the lines, the Weyl group of $\mathbf{E_6}$. The two fundamental roles played by $W(E_6)$ (respectively by its simple normal subgroup of index two, $G_{25,920}$) are:

- $W(E_6)$ (resp. $G_{25,920}$) is a reflection (resp. unitary reflection) group in $\mathbb{P}^5$ (resp. $\mathbb{P}^4$ and $\mathbb{P}^3$).
- $W(E_6)$ (resp. $G_{25,920}$) is the Galois group of an algebraic equation, namely the algebraic equation for the 27 lines on the cubic surface (resp. the algebraic equation which results after extracting the discriminant of the above).

We will consider the second role in this chapter and the first in the following chapters. Of course the two are interwoven, and in fact the latter problem is ***reduced*** to the former, and it is the former (a form problem) which can be solved (by transcendental methods).

In the first section we recall the normal forms for equations of cubic surfaces and the configurations related to the 27 lines. In the second we recall the general method developed in the last century to solve algebraic equations with simple Galois groups, giving the standard example of the quintic as developed in [Kl3]. In the third section we then recall the actions of $G_{25,920}$ on $\mathbb{P}^3$ and $\mathbb{P}^4$, and the invariants, etc., of these actions. Finally, in the last section we discuss the equation for the 27 lines, an equation with Galois group $W(E_6)$. There are two approaches to solving the equation. The first, due to Klein and Jordan, and carried out by Maschke and Burkhardt, uses foremost the representation of $G_{25,920}$ in $\mathbb{P}^3$ and its invariants. The second, due to Coble, uses the representation in $\mathbb{P}^4$ and reduces the problem to the form problem for this group, and then to a ***special*** form problem, which is then a problem about the invariant of fourth degree J_4, the topic of the next chapter. We should remark that as far as solving the equation is concerned, all this work is more or less unnecessary. In fact, one has a standard method for giving the solution of ***any*** equation in terms of theta functions, as shown in the appendix to Mumford's Theta III. But the work is worth it as far as understanding the geometry of the situation is concerned.

4.1 Cubic surfaces

In this section we introduce a key configuration for this and the following examples. Although of course very well-known, we review many of the facts about the 27 lines on a general cubic surface.

We begin with the classical approach, in terms of cubic forms, and then proceed to the more modern approach, in which the cubic surface is considered as a del Pezzo surface of degree 6.

4.1.1 The classical approach

Consider a smooth cubic surface S in $\mathbb{P}^3(\mathbb{C})$, given by the vanishing of a cubic form f. That S should only contain a finite number of lines is a dimension count, as follows. An n-dimensional variety V in $\mathbb{P}^N$ of degree d will meet, by definition, a (generic) $\mathbb{P}^{N-n}$ in exactly d points. Hence, if a $\mathbb{P}^{N-n}$ contains $d+1$ or more points of V, it lies entirely in V, so if $d+1 = \dim Gras(\mathbb{P}^{N-n}, \mathbb{P}^N) = n(N-n+1)$, there should be finitely many $\mathbb{P}^{N-n}$'s lying on V. In our case $N = 3$, $N-n = 1$, $d = 3$, $d+1 = 4 = \dim Gras(\mathbb{P}^1, \mathbb{P}^3)$. However the determination of the precise number is much more delicate and was done for the first time by George Salmon in 1849, in response to a letter from Arthur Cayley. The argument he used is reproduced for example in [Sal], and is purely algebraic. For the convenience of the reader we sketch this briefly.

Consider hyperplane sections H_t passing through one of the lines. Since $H_t \cap S$ is a third degree plane curve, which already contains a line, it consists generically of a quadric and a line. Since this intersection contains two double points, it follows that ***every*** such plane is a bitangent, i.e., tangent to S in two points. For a finite number of hyperplanes H_t this intersection degenerates into three lines, as illustrated in Figure 4.1.

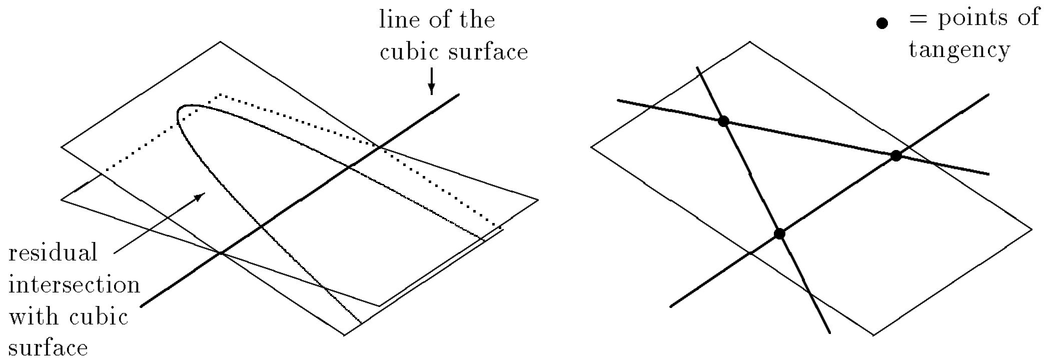


Figure 4.1: (a) generic (transversal) section (b) tangent section

Such planes H_t which are tangent to S at three points (and contain three lines) are called accordingly ***tritangent planes.*** Now fix a line L on the cubic surface given in local coordinates by $x_3 = x_4 = 0$, say. Then the equation $f = 0$ can be written $x_3U + x_4V = 0$ with U, V quadratic. To find the tritangent planes one puts $x_3 = \mu x_4$ into this relation, divides by x_4 and forms the discriminant of the ensuing equation. Viewing this as an equation in μ, one sees easily that it has degree 5, i.e., there are five values of μ for which the corresponding plane is a tritangent, or in other words the given line lies in five tritangent planes. Now count: starting with a given tritangent plane, it meets $4{\cdot}3=12$ other tritangents, in each of which there are two other lines, which gives $12{\cdot}2+3=27$ lines on a smooth cubic surface. Since each line is contained in five tritangents, this leads to $27{\cdot}5\div3=45$ tritangent planes to the cubic surface S. In the book [Sal] several other proofs of the magic number 27 are given.

We consider now the group of permutations, $\mathrm{Aut}(\mathcal{L})$, of the 27 lines (or of the 45 planes), by which we mean the permutations of the lines preserving the intersection behavior of the lines. For this it is useful to consider the famous double sixes and the notation for the 27 lines introduced by Schläfli. A ***double six*** is an array

$$N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

of 12 of the 27 lines with the property that two of these 12 meet if and only if they are in different rows and columns. (This notation distinguishes this particular set of 12, although any such double six is equivalent to it under Aut($\mathcal{L}$)). The other lines are given by the $\binom{6}{2}$=15 $c_{ij} = a_ib_j \cap a_jb_i$, where a_ib_j denotes the tritangent spanned by those two lines. There are 36 double sixes, namely the N above, 15 N_{ij} and 20 N_{ijk}:

$$\begin{aligned} N_{ij} &= \begin{bmatrix} a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\ a_j & b_j & c_{ik} & c_{il} & c_{im} & c_{in} \end{bmatrix} \\ N_{ijk} &= \begin{bmatrix} a_i & a_j & a_k & c_{mn} & c_{ln} & c_{lm} \\ c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n \end{bmatrix} \end{aligned} \tag{4.1}$$

Since a double six describes, by definition, the intersection behavior of the lines, we see immediately that Σ_6 (the symmetric group on six letters) acts by permutations on a double six and a $\mathbb{Z}_2$ acts by exchanging rows. Since there are 36 double sixes, we see $|\text{Aut}(\mathcal{L})| = |\Sigma_6| \cdot 2 \cdot 36 = 51{,}840$. In fact Aut($\mathcal{L}$) is nothing but the Weyl group $W(E_6)$, and the 36 double sixes correspond to the positive roots. The 27 lines correspond to the 27 fundamental weights of E_6, and the many other sets of objects (lines, tritangents, etc.)correspond to natural sets of objects (roots, weights, etc.) of E_6. In the remainder of this section we will consider the sets of objects assoicated with the 27 lnes. Many of these will be translated below into the $W(E_6)$ formulation, in particular in Chapter 6. Consider two double sixes; a natural question arising here is: how many lines do they have in common? The answer is twofold:

- either: four (like a_1, a_2, b_1, b_2, which N and N_{12} have in common) which have the property of lying in pairs in planes, the pairs being however mutually disjoint;

- or: six (like a_1, a_2, a_3, b_1, b_2, b_3, which N and N_{123} have in common) which form two triples.

Following Sylvester one speaks accordingly of ***syzygetic*** and ***azygetic*** pairs of double sixes. A given double-six is syzygetic to 15 others and azygetic to 20 others. A pair of azygetic double sixes form through the 12 lines they do ***not*** have in common a third double-six, which is azygetic to both. There are 120 such triads of azygetic double sixes, and 36·15÷2=270 pairs of syzygetic double sixes.

A further geometric curiosity of the 45 planes is the following. Take two of the tritangents which do not meet in a line on the cubic surface, say α_1, α_2 (for example $a_1b_2c_{12}$ and $a_3b_4c_{34}$). These two planes determine three other planes, denoted β_1, β_2, β_3, by the property of containing each a line of α_1 and of α_2 (in the example above, $a_1b_4c_{14}$, $b_2a_3c_{23}$, $c_{12}c_{34}c_{56}$). The third line in each of β_i which is not one of α_i all lie in a common tritangent (for example here $c_{14}c_{23}c_{56}$). Then this is a unique third tritangent denoted α_3. The set of 6 tritangents $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ have the special property that the nine lines $\alpha_\mu \cap \beta_\nu$ together with a point (19 conditions) determine S. Such a set of six tritangents is called a ***trihedral pair***, and there are 120 such; this implies that the equation of the cubic surface S can be written in 120 ways as

$$y_1y_2y_3 + z_1z_2z_3 = 0.$$

Furthermore there are 40 ***triads (triples) of trihedral pairs***, such that each such triad contains all 27 lines. The α's meet the β's in nine lines on the cubic surface and vica versa. One finds the following types:

Table 4.1: Loci assoicated with the 27 lines on a smooth cubic surface

# objects	description of the objects
27	lines on a cubic surface
135	intersection points of two of the lines
216	pairs of skew lines
36	double sixes
45	tritangents
120	trihedral pairs, set of six tritangents containing nine lines
40	triples of trihedral pairs, set of 18 tritangents containing all 27 lines
120	triads of azygetic double sixes
270	pairs of syzygetic double sixes

$$\begin{array}{|ccc|ccc|ccc|}\hline \multicolumn{3}{|c|}{(20)} & \multicolumn{3}{c|}{(10)} & \multicolumn{3}{c|}{(90)} \\ \hline a_i & b_j & c_{ij} & c_{il} & c_{jm} & c_{kn} & a_i & b_j & c_{ij} \\ b_k & c_{jk} & a_j & c_{mn} & c_{ik} & c_{lj} & b_l & a_k & c_{kl} \\ c_{ik} & a_k & b_i & c_{jk} & c_{ln} & c_{im} & c_{il} & c_{jk} & c_{mn} \\ \hline \end{array}, \tag{4.2}$$

the rows (of each box) giving α_1, α_2, α_3 and the columns giving β_1, β_2, β_3. This configuration is in some sense complementary to the double sixes: *Starting with nine lines lying in such a trihedral pair, the remaining 18 lines form a unique azygetic triple of double sixes and conversely, the nine lines not contained in a given azygetic triple of double sixes always lie in a trihedral pair.* We sum up the configuration in Table 4.1.

4.1.2 Equations

As far as the equation defining a cubic surface is concerned, we have the following result, claimed by Sylvester (1851) and Steiner, and proved by Clebsch (1861); it is the so-called *pentrahedral form* of the cubic.

Theorem 4.1.1 *A general cubic form $F(x_0 : x_1 : x_2 : x_3)$ can be put, in a* unique *way, in the form*

$$a_1y_1^3 + a_2y_2^3 + a_3y_3^3 + a_4y_4^3 + a_5y_5^3 = 0,$$

where the coordinates y_i are linear in $(x_0 : \cdots : x_3)$ and satisfy

$$y_1 + \ldots + y_5 = 0.$$

The five planes $y_i = 0$ are the faces of a *Sylvester pentahedron*; they meet in pairs in ten lines and three at a time in ten points. These ten lines are contained in the Hessian of the cubic surface S (a quartic, see section B.5.3.2), and the ten points are *double points* of that Hessian. Viewing these ten points as hyperplanes (in the dual $\mathbb{P}^3$), the product gives a polynomial of degree ten, whose vanishing defines the ten points. Clebsch's proof of 4.1.1 was by showing that the degree ten equation has a resolvent of degree five (see Lemma 4.2.2 below), which can be solved by the methods described in the next section (see in particular Theorem 4.2.9 and Figures 4.6, 4.7 and 4.10 below). The five solutions of this equation then determine five linear forms y_i in the x_i, and the uniqueness[1] of the representation above is relatively clear from the construction. This form of the cubic equation is very convenient for the ***invariant theory*** (see section B.4 in the Appendix). We also mention that in the appendix we describe the Hessian, Steinerian and dual varieties of a cubic surface. Also there the notion of invariants of the cubic form is introduced, and the known invariants are listed.

A second special form of the equation, the ***hexahedral form***, is given by the ***polar hexagons***:

Theorem 4.1.2 *A general cubic form F can be put in a four-dimensional family of forms:*

$$z_1^3 + \cdots + z_6^3 = 0,$$

where the z_i are linear in the x_i.

Cremona (Math. Ann. XIII) recognized that there are 36 special equations of the form 4.1.2 – one associated with each double six. Indeed, the 15 lines ***not*** in the given double six lie in 15 tritangent planes, and from these one can construct ten trihedral pairs; the ten pairs of intersection points of the trihedra form the pairs of ***opposite*** vertices of the hexagon (six planes meet in $\binom{6}{2} = 15$ lines and $\binom{6}{3} = 20$ points). Furthermore, from the hexahedral equation one can get the pentahedral equation.

Proposition 4.1.3 (Cremona, 1840) *In each of the 36 hexagons associated with any of the double sixes of the cubic surface S, there is a space cubic curve inscribed, all of which have the same five ocsulating planes; these form the pentahedron of 4.1.1.*

Cremona also noted that for a cubic in hexahedral form, one can explicitly write down the equations for the 27 lines and 45 tritangents. We will return to this below. It is possible to pass from either of the above equations to the other; this is described in [Sal], §310 (p. 403). The hexahedral form determines immediately the pentahedral form by the previous proposition, while the determination of the hexahedral form from a given pentahedral form requires the solution of a sextic equation. However, there is no purely (rationally) algebraic method of passing from a given quatenary cubic F to the pentahedral or hexahedral form, which follows from the fact that the Galois group of the problem is simple; indeed any algebraic reduction to the form 4.1.1 would imply solvability of the Galois group; this actually occurs for special cubic surfaces, but not in general.

Finally we briefly describe the "equation of 27^{th} degree which determines the 27 lines" which will be discussed extensively below; see in particular section 4.3 for this. It was already known to Cayley that one can write the set of 27 lines as the intersection of a covariant of degree nine of the cubic with the cubic surface S, as described in (B.51) of the appendix. This is an equation of degree 27 in the variables of $\mathbb{P}^3$, and applying elimination theory reduces this (but not in a covariant manner) to a degree 27 equation in a single variable. It almost goes without saying that the Galois group of this equation is $\mathrm{Aut}(\boldsymbol{\mathcal{L}})$. It can be reduced to $G_{25,920}$ by extracting the square root of the discriminant (see, for example, [Kl1]).

4.1.3 Special cubic surfaces

There are two particular cubic surfaces which are very special and easy to study. We will do that in this section, where we will use several general results on cubic surfaces proved or stated later, to give a rather complete discussion of them. Let us list several aspects, about which one would generaly like to know as much as possible for any given cubic surface S:

1. the pentahedral equation 4.1.1;

[1] assuming F is generic, $\mathrm{Aut}(S) = \{1\}$. For cubics with non-trivial automorphisms, for example if $a_i = a_j$, uniqueness is lost, but there will still be finitely many pentahedral forms representing a given F.

2. the hexahedral form 4.1.2;
3. the 36 sets of six points in the plane which are defined by the double sixes of S;
4. invariants and linear covariants of S (B.49) and (B.50);
5. the equations of the lines and tritangents;
6. the Hessian variety, a symmetroid quartic surface (B.5.11);
7. the dual variety (B.37).

The equations were discussed above. The sets of six points are considered in general in the next section. Finding equations for the lines and tritangents is a kind of unifying theme of the rest of the book and occurs repeatedly from now on. The symmetroids are discussed in the appendix on quartic surfaces, and the equation of the dual variety is described in the appendix on cubic forms. As far as it is feasible we will discuss these issues for the two special cubic surfaces. These surfaces each have a comparitively large automorphism group, the symmetric groups Σ_4 and Σ_5, respectively, and correspondingly, each is particularly symmetric. A general cubic surface will not be as symmetric as these are, a fact the reader should bear in mind.

The Cayley cubic $\mathcal{C}$. There is a unique cubic surface with four nodes, usually called the Cayley cubic surface, which we will denote by $\mathcal{C}$. If one takes the four vertices

$$(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$$

to be those nodes (this does not restrict generality as any four points in $\mathbb{P}^3$ are equivalent to these under $PGL(4)$), the equation can be written

$$\frac{1}{u_0}+\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}=0, \text{ or } u_0u_1u_2+u_0u_1u_3+u_0u_2u_3+u_1u_2u_3=0. \tag{4.3}$$

First we describe the pentahedral equation for $\mathcal{C}$.

Proposition 4.1.4 *The pentahedral form of the Cayley cubic (4.3) is*

$$\mathcal{C}=\{y_1^3+y_2^3+y_3^3+y_4^3+\frac{1}{4}y_5^3=0\},\quad \sum y_i=0.$$

Proof: We recall from the discussion of the Segre cubic $\mathcal{S}_3$ that there were special kinds of hyperplane sections; the sections x_i+x_j cut $\mathcal{S}_3$ in three $\mathbb{P}^2$'s, the sections $x_i=0$ cut out the diagonal surface, while the other sections of interest were the 15

$$\mathcal{S}_3\cap\{\mathcal{T}_{ij}=0\}, \tag{4.4}$$

where $\mathcal{T}_{ij}=x_i-x_j$ contains four of the nodes, for example $\mathcal{T}_{56}=x_5-x_6$ contains

$$(1,-1,-1,-1,1,1),\ (1,1,1,-1,-1,-1),\ (1,1,-1,1,-1,-1)\ \text{ and } (1,-1,1,1,-1,-1).$$

Since as is easily checked the cubic surface (4.4) is irreducible, it has four nodes and so does $\mathcal{C}$. Since $\mathcal{C}$ is the unique cubic surface with four nodes, this hyperplane section is isomorphic to $\mathcal{C}$. From this we can determine the pentahedral equation: we have $\sum_1^6 x_i=0,\ x_5-x_6=0$, so the equation for $\mathcal{S}_3$ becomes:

$$x_1^3+x_2^3+x_3^3+x_4^3+2x_5^3=0,\quad 2x_5=-x_1-x_2-x_3-x_4.$$

So if we set:

$$y_i = x_i, \ i = 1, \dots, 4, \quad y_5 = 2x_5, \tag{4.5}$$

then we have $\sum_1^5 y_i = 0$ and the equation for $\mathcal{C}$ becomes

$$x_1^3 + \cdots + x_4^3 + 2x_5^3 = y_1^3 + \cdots + y_4^3 + \frac{1}{4}y_5^3 = 0,$$

which is the desired pentahedral form. $\square$

Since the cubic surface $\mathcal{C}$ is *singular*, there are not 27 lines, 45 tritangents and 36 double sixes. As to the double sixes: since $\mathcal{C}$ is a section of the Segre cubic $\mathcal{S}_3$, we get immediately its hexahedral form:

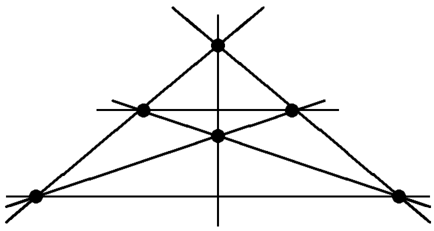
Proposition 4.1.5 *The 15 hexahedral equations of $\mathcal{C}$ are given by the Σ_6-orbit of*

$$(\overline{a}, \dots, \overline{f}) = (0, 0, 0, 0, 1, -1).$$

Proof: We have the coordinates $(a, \dots, f) = (x_1, \dots, x_6)$, and the equation of $\mathcal{T}_{56}$ defines the $(\overline{a}, \dots, \overline{f})$ as stated. $\square$

To count the lines, let us begin by describing the six points in the plane giving rise to $\mathcal{C}$.

Proposition 4.1.6 *The six points in $\mathbb{P}^2$ determining $\mathcal{C}$ are the six intersection points of a complete quadralateral, i.e., as pictured:*



The thin lines are the three diagonals *of the quadralateral. The set of six points is rigid; it cannot be deformed, hence defines a unique set of six points.*

Proof: Blowing up the six points, each edge of the quadralateral contains three points, hence becomes a -2-curve and is blown down to an ordinary node, so the corresponding cubic surface has four nodes. As such a surface is unique, it is (isomorphic to) $\mathcal{C}$. $\square$

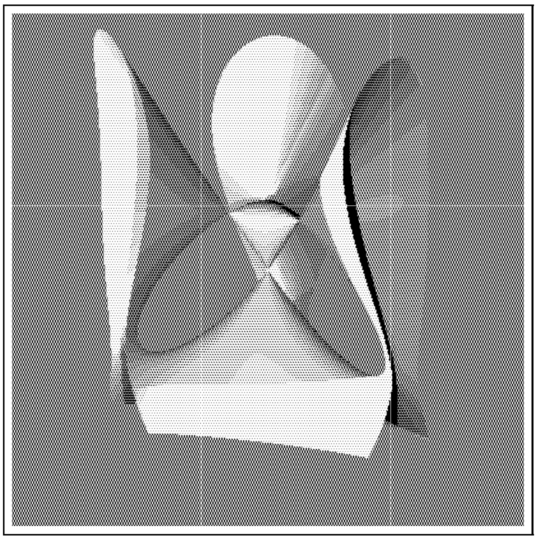
Now we can find all lines on $\mathcal{C}$. They are the images of the six points, the lines joining the six points in twos, of which there are now only three instead of 15, the diagonals of the quadlateral, and the conics on five points, all of which coincide with sets of two of the four lines (and which get blown down on the surface). This establishes the first statement of

Proposition 4.1.7 *The Cayley cubic $\mathcal{C}$ contains nine lines and 11 tritangents. The latter are given by:*

$$\begin{array}{lll} y_i + y_j, & 1 \le i < j \le 4 & (6) \\ 2y_i + y_5 & & (4) \\ \qquad y_5 & & (1). \end{array}$$

Proof: The 15 Segre planes on $\mathcal{S}_3$, intersected with $\mathcal{C}$, give five lines. These 15 planes are cut out by the 15 hyperplanes $x_i + x_j, 1 \le i < j \le 6$, which from $2x_5 = y_5$ become the 11 listed planes. That there are no further tritangents follows from (4.14) below and the hexahedral form 4.1.5 for $\mathcal{C}$. Indeed,

Figure 4.2: The Cayley cubic surface



The affine equation used here is

$$(1-3x-3y-3z)(xy+xz+yz)+6xyz=0.$$

These affine coordinates are derived by sending the coordinate tetrahedron $\Delta=\{u_0u_1u_2u_3=0\}$ to the simplex $\Delta'=\{xyz(x+y+z-1/3)=0\}$. As a plane at infinity one chooses $v=3(u_0+u_1+u_2+2u_3)$, so that, in terms of the new coordinates u_0,u_1,u_2,v, we have $u_3=\frac{1}{2}(1/3v-u_0-u_1-u_2)$, and this is inserted in the homogenous equation for $\mathcal{C}$. We get

$$\mathcal{C}=\{u_0u_1u_2+(u_0u_1+u_0u_2+u_1u_2)\frac{1}{2}(\frac{1}{3}v-u_0-u_1-u_2)=0\}.$$

Now dehomogenizing by dividing by v^3 and setting $x=u_0/v,\ y=u_1/v, z=u_2/v$, we get the affine equation

$$\begin{aligned}\mathcal{C} &= \left\{xyz+\frac{1}{2}(xy+yz+xz)(\frac{1}{3}-x-y-z)=0\right\}\\ &= \left\{\frac{1}{6}(6xyz+(xy+yz+xz)(1-3x-3y-3z)=0)\right\}.\end{aligned}$$

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in this case $d_2=1$, while $\overline{a}=\overline{b}=\overline{c}=\overline{d}=0,\ \overline{e}=1,\overline{f}=-1$. So the equations (4.14) become, for example

$$(\overline{e}\overline{f}-1)(e+f)\equiv(-2)(x_5+x_6)=-4x_5=-2y_5,$$

so this plane has already been acounted for, and similarly for the others. □

Remark 4.1.8 Actually, in the case of this singular cubic, the Galois group of the equation for the 27 lines reduces from the Weyl group of E_6 to the tetrahedral group, hence is solvable, so at any rate one should be able to calculate these tritangents.

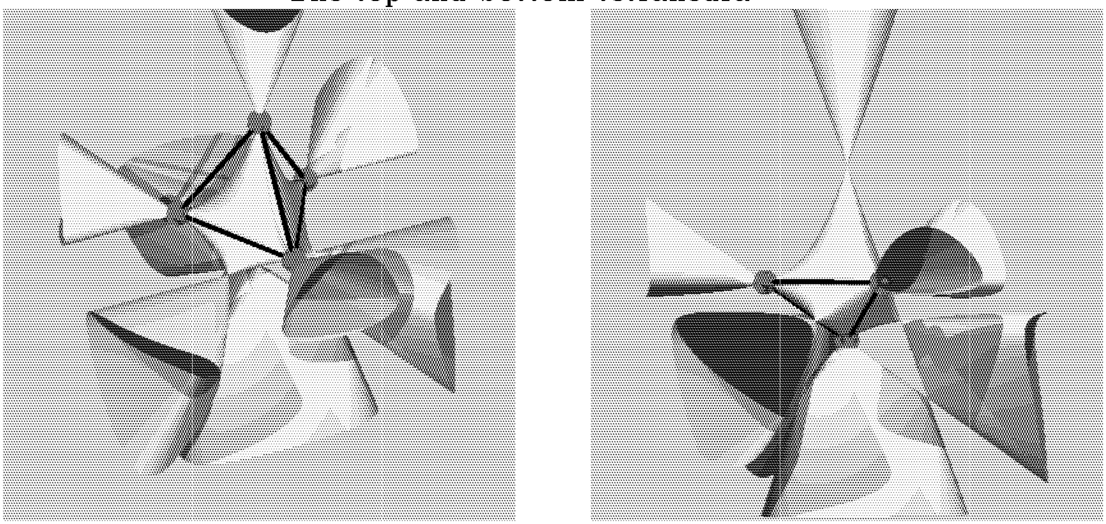
Since there are only nine lines there of course no double sixes. Now we can easily determine which lines are contained in which tritangents. The nine lines are:

$$\begin{array}{ll}(3) & y_5=0=y_i+y_j=y_k+y_l,\ \ \{i,j,k,l\}=\{1,2,3,4\},\\ (6) & 2y_i+y_5=2y_j+y_5,\ 1\le i<j\le 4.\end{array}$$

The first three are contained in three tritangents apiece, while the latter six lines are contained each in only two of the tritangents. Note that the latter six lines are just the edges of the coordinate tetrahedron. To visualize the other three, we refer to the second picture in Figure 4.3. These three lines are horizontal and join the three lower local minima of $\mathcal{C}$.

The Hessian variety of $\mathcal{C}$, as for any cubic surface, is a quartic with ten nodes, the ten nodes being the ten vertices of the Sylvester pentahedron. The Sylvester pentahedron of the cubic surface intersects the Hessian in the union of ten lines on that quartic. A picture for the Cayley cubic is given in Figure 4.3. This quartic has in fact 14 nodes instead of just ten, which can be seen as follows. In the following set of pictures, two tetrahedra have been drawn into the view of the Hessian.

The top and bottom tetrahedra



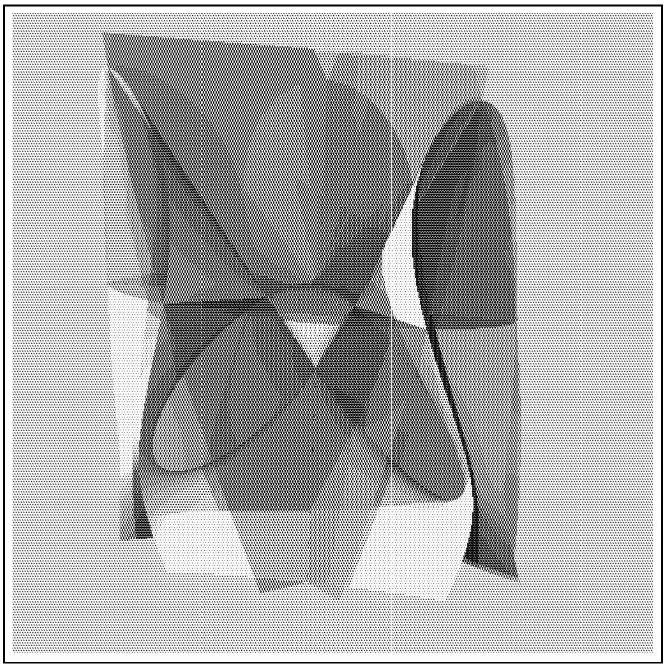
Note that the central teepee forms a tetrahedron a "top" tetrahedron. Now locate the bottom face of that tetrahedron; at the center of that face is a node, connecting to the bottom branch of the surface, which goes vertically downwards. This node and the three other vertices of the face form another tetrahedron, a "bottom" tetrahedron. The three lower branches have two nodes apiece, one at the center of each face of the bottom tetrahedron, and one on the bottom branch of the surface. Hence there are three planes[2], horizontal in the picture, each of which contains three nodes, accounting for nine nodes. In addition there is the node joining the bottom tetrahedron with the bottom branch, and the node joining the top tetrahedron with the top branch, so we see in all 11 nodes. There are three further nodes in the plane at infinity, so this quartic has 14 ordinary double points (note that the two parallel planes of the Sylvester pentahedron meet in the plane at infinity, and this two-fold line meets each of the other three planes of the Sylvester pentahedron in a point which is a vertex of the Sylvester pentahedron, i.e., a point where three of the planes meet, hence a node of the Hessian). Of course the four additional nodes (compared with the Hessian of a smooth cubic surface) come from the nodes of the Cayley cubic, and with some imagination, one can superimpose the center tetrahedron of the Cayley cubic over the top tetrahedron of the Hessian and see the four additional nodes (the pictures are not at exactly the same scale and angle, but roughly). Looking at the picture of the Hessian with the planes of the Sylvester pentahedron, the four additional nodes are precisely those not lying on one of these five planes – these four points are the vertices of the bottom tetrahedron.

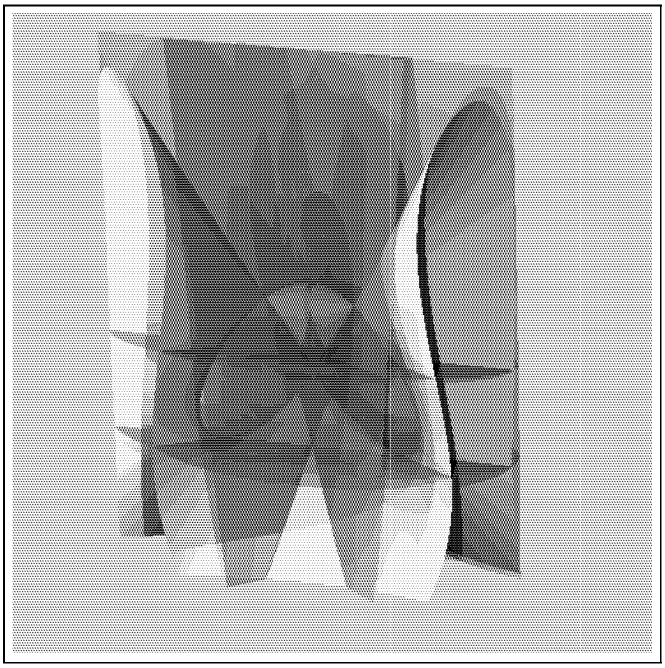
Note that in the picture with the Sylvester pentahedron, one of the five planes is *head on*, and one only sees a line. That plane contains four of the nodes which are finite. Finally, we describe the transition from pentahedral to tetrahedral coordinates. As we have seen, the tetrahedron of reference

[2]Two of these planes are the horizontal planes of the Sylvester pentahedron. The third, lying above the other two, contains the three vertices of the common face of the top and bottom tetrahedron.

Figure 4.3: The Cayley cubic, its Hessian and the Sylvester pentahedron

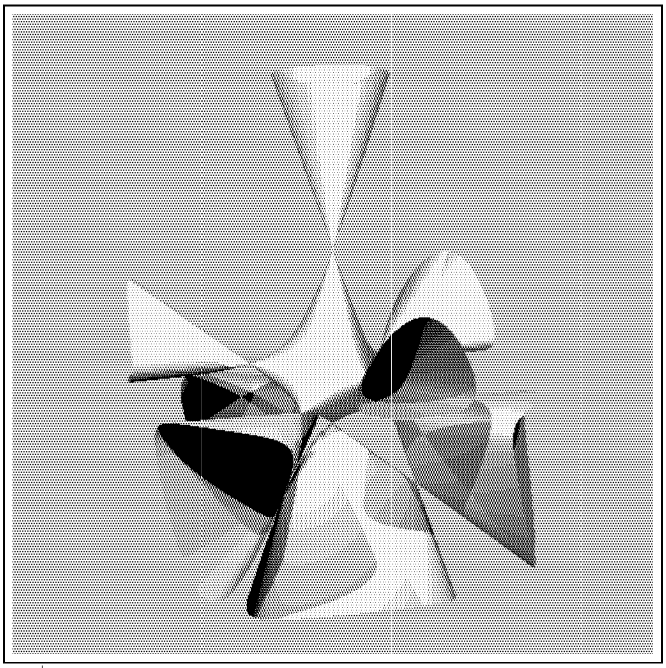
The Cayley cubic, the coordinate tetrahedron and the Sylvester pentahedron





The first picture depicts the coordinate tetrahedron, which consists of four tritangent planes. The second figure shows the Sylvester pentahedron of the Cayley cubic.

The Hessian of the Cayley cubic and the Sylvester pentahedron



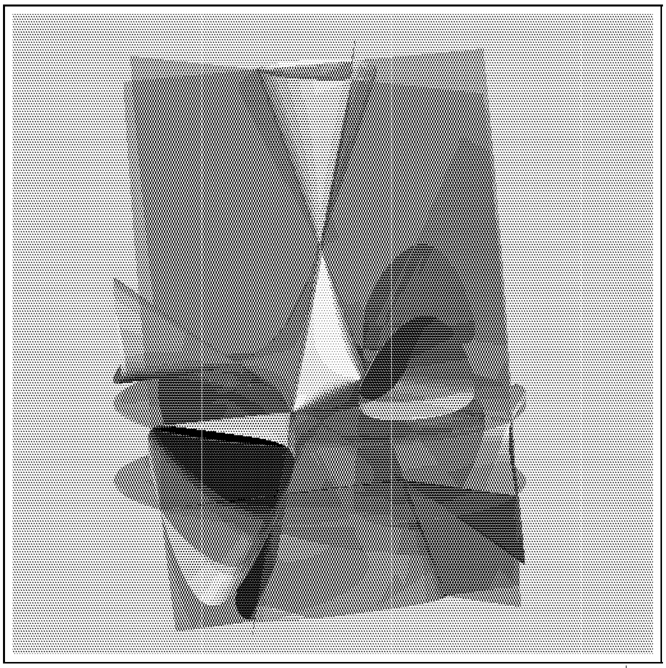
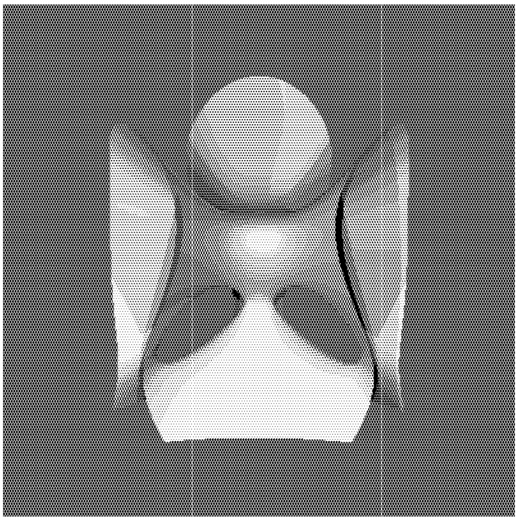


Figure 4.4: The Clebsch diagonal cubic surface



The affine equation for the Clebsch cubic $\mathfrak{C}$ is derived as follows. Inserting the expressions in the u_i of (4.10) into the equation (4.8) for $\mathfrak{C}$, we get an expression in the u_i. Then divide by v^3, where $v = 2(u_0+u_1+u_2+4u_3)$ is the plane at infinity. The result is

$$\begin{aligned}\mathfrak{C} \;=\; & \{81(x^3+y^3+z^3) - 189(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2)+ \\ & 54xyz + 126(xy+xz+yz) - 9(x^2+y^2+z^2) - 9(x+y+z)+1=0\}.\end{aligned}$$

---

is given by $\Delta = \{u_0u_1u_2u_3 = 0\}$. In the y_i coordinates, we take the four tritangents $2y_i + y_5$ of Proposition 4.1.7, i.e., we set

$$\begin{aligned} u_0 &= 2y_1 + y_5, \qquad & u_1 &= 2y_2 + y_5, \\ u_2 &= 2y_3 + y_5, \qquad & u_3 &= 2y_4 + y_5, \end{aligned} \tag{4.6}$$

and a direct verification shows that

$$u_0u_1u_2 + u_0u_1u_3 + u_0u_2u_3 + u_1u_2u_3 = \frac{8}{3}\left(\sum_1^4 y_i^3 + \frac{1}{4}y_5^3\right).$$

Inverting the equations we have

$$\begin{aligned} y_5 &= \tfrac{1}{2}(u_0+u_1+u_2+u_3), \quad & y_1 &= -\tfrac{1}{4}(-u_0+u_1+u_2+u_3), \\ y_2 &= -\tfrac{1}{4}(u_0-u_1+u_2+u_3), \quad & y_3 &= -\tfrac{1}{4}(u_0+u_1-u_2+u_3), \\ & & y_4 &= -\tfrac{1}{4}(u_0+u_1+u_2-u_3), \end{aligned} \tag{4.7}$$

and these are the equations for the Sylvester pentahedron in tetrahedral coordinates. The four planes of (4.6) are tritangents but the latter four of (4.7) are not, while the plane $y_5 = 0$ does turn out to be a tritangent. These sets of planes are depicted in Figure 4.3.

If we take the affinisation given by $u_3 = 1/2(1/3 - x - y - z)$, then the equations are:

$$\begin{aligned} y_1 &= \tfrac{1}{6}(-9x+3y+3z+1), \quad & y_2 &= \tfrac{1}{6}(3x-9y+3z+1), \\ y_3 &= \tfrac{1}{6}(3x+3y-9z+1), \quad & y_4 &= \tfrac{1}{6}(9x+9y+9z-1), \\ & & y_5 &= \tfrac{1}{3}(1+3x+3y+3z). \end{aligned}$$

Of these, the plane $(1+3x+3y+3z)=0$ is the special tritangent mentioned above. The other pentahedral planes are, as one can see in the picture, not tritangents. In the picture with the Hessian we have instead used $u_3=1/2(2/5-x-y-z)$; then the equations are

$$y_1=\tfrac{1}{40}(15x-5y-5z-2),\quad y_2=\tfrac{1}{40}(-5x+15y-5z-2),$$
$$y_3=\tfrac{1}{40}(-5x-5y+15z-2),\quad y_4=\tfrac{1}{40}(-15x-15y-15z+2),$$
$$y_5=\tfrac{1}{20}(2+5x+5y+5z).$$

The Clebsch diagonal cubic $\mathfrak{C}$

The other very famous cubic surface is given by the following pentahedral equation:

$$\mathfrak{C}=\{y_1^3+\cdots+y_5^3=0=\sum y_i\};\tag{4.8}$$

it is depicted in Figure 4.4. This surface happens to be given in pentahedral form. Recall that $\mathfrak{C}$ is isomorphic to the hyperplane sections $x_i=0$ of the Segre cubic $\mathcal{S}_3$. As $\mathfrak{C}$ is a section of the Segre cubic, again we get immediately the hexahedral form.

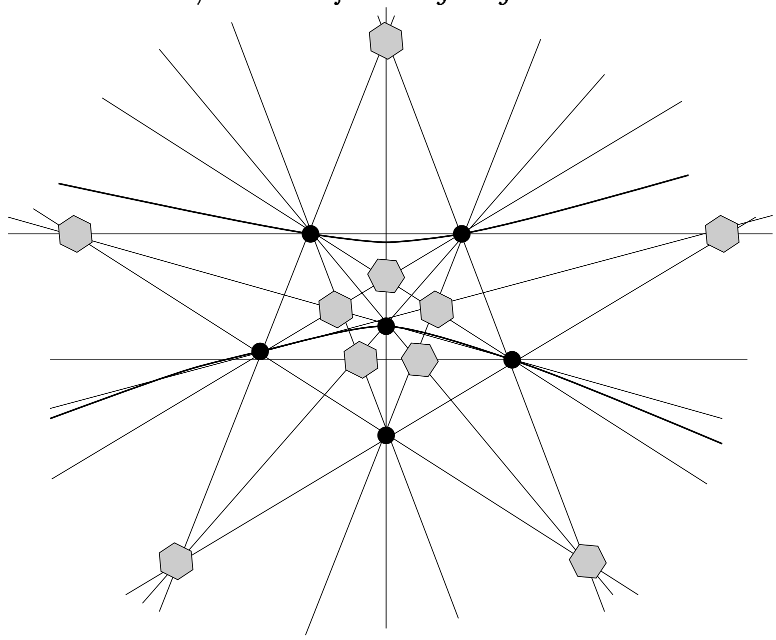
Proposition 4.1.9 *The 36 hexahedral forms for $\mathfrak{C}$ are given by the $W(E_6)$-orbit of*

$$(\overline{a},\ldots,\overline{f})=(1,\ldots,1,-5).$$

Proof: In the x_i the equation is $\sum_1^6 x_i^3=0$, while $x_6=0$ on the one hand and $\overline{a}+\cdots+\overline{f}=0$ on the other. It follows that $\overline{a}=\cdots=\overline{e}$, and hence $(\overline{a},\ldots,\overline{f})=(1,\ldots,1,-5)$, as stated. □

The Clebsch cubic $\mathfrak{C}$ is smooth, hence contains 27 lines, has 45 tritangents, 36 double sixes, etc. But these 27 lines are in very special position. To begin with, the set of six points in the plane must admit an action of Σ_5. There is a unique such set, given as follows:

Proposition 4.1.10 *Any set of six ponts in $\mathbb{P}^2$ giving rise to $\mathfrak{C}$ is projectively equivalent to the vertices of a pentagram, and the center, as in the following diagram.*



The ten points indicated by the hexagons are Eckard points, *which are three-fold points of the arrangement; the six points are five-fold points of the arrangement. One of the conics through five of the six points is also drawn.*

Proof: This is the only Σ_5-invariant set of six points[3], and indeed $\mathfrak{C}$ is the unique cubic surface whose automorphism group is Σ_5. $\square$

Given three of the c_{ij} in $\mathbb{P}^2$, the triangle they form in $\mathbb{P}^2$ becomes a triangle on the cubic surface, i.e., is the intersection of the cubic surface with a tritangent. In case c_{ij}, c_{kl}, c_{mn} all meet ***in a single point***, (that is the triangle shrinks to a point), one refers to the point as an ***Eckard point*** of the surface[4], and the tritangent $< c_{ij}c_{kl}c_{mn} >$ is called an ***Eckard plane.*** From the diagram above we see that $\mathfrak{C}$ has ten such Eckard points, and again, $\mathfrak{C}$ is the unique cubic surface with ten Eckard points. Note that these points occur, when, writing the cubic surface in pentahedral form, two or more of the coefficients coincide; for $\mathfrak{C}$ there are five coefficients which coincide, hence $\binom{5}{2} = 10$ Eckard points.

Corollary 4.1.11 *$\mathfrak{C}$ is the unique cubic surface with ten Eckard points.*

Before turning to the equations for the lines and tritangents, we pause to explain the name "diagonal" cubic. The five planes $y_i = 0$ of the Sylvester pentahedron meet in ten lines, which meet three at a time in ten points. In each such pentahedral plane, the intersection with the other four form a complete quadralateral as in Proposition 4.1.6. The six intersection points are all Eckard points, so the ten vertices of the Sylvester pentahedron lie on the ten Eckard planes. The consequence of this is that the diagonals of the quadralaterals in each pentahedral plane are lines on the cubic $\mathfrak{C}$, as these lines join two Eckard points, hence meet six of the lines of the surface (and are consequently themselves lines on the surface).

Now we turn to the equations for the 27 lines and 45 tritangents, which in this case are easily derived, as we have the hexahedral form for the surface.

Proposition 4.1.12 *15 tritangents of $\mathfrak{C}$ are given by the equations*

$$0 = y_i + y_j,\ 1 \le i < j \le 5 \quad (10), \qquad y_i = 0,\ i = 1, \dots, 5 \quad (5).$$

Of these, the first ten are the Eckard planes of $\mathfrak{C}$. These 15 tritangents cut $\mathfrak{C}$ in 15 lines, namely the 15 lines complementary to the double six on $\mathfrak{C}$ used to get the hexahedral equation. The remaining tritangents are given by equations defined over $\mathbb{Q}(\sqrt{5})$, as described in (4.14). They are as follows:

$$\begin{array}{llll}
\alpha_{ijkl} & 0 = (1 - 3\sqrt{5})(y_i + y_j) - (1 + 3\sqrt{5})(y_k + y_l), & i \neq j \neq k \neq l \neq 5 & (6) \\
\beta_{ijk} & 0 = (1 - 3\sqrt{5})(y_i + y_j) - (3\sqrt{5} - 5)(y_k + y_5), & i \neq j \neq k \neq 5 & (12) \\
\gamma_{ijk} & 0 = (-5 - 3\sqrt{5})(y_i + y_5) - (1 + 3\sqrt{5})(y_j + y_k), & i \neq j \neq k \neq 5 & (12) \\
\delta_{ij} & 0 = (-5 - 3\sqrt{5})(y_i + y_5) - (-5 + 3\sqrt{5})(y_i + y_5), & i \neq j \neq 5 & (6).
\end{array}$$

The 27 lines are given by:

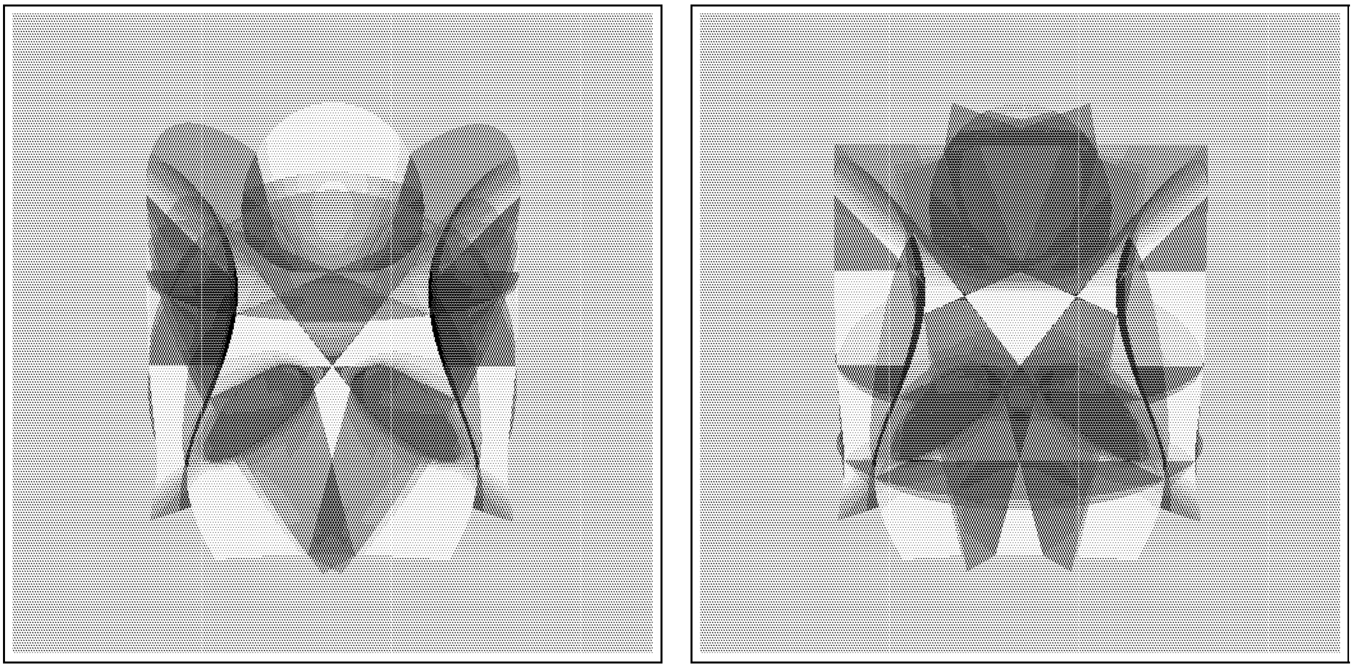
$$\begin{array}{lll}
0 = y_i + y_j = y_k + y_l, & i \neq j \neq k \neq l \neq 5 & (3) \\
0 = y_i + y_j = y_k + y_5, & i \neq j \neq k \neq 5 & (12) \\
0 = \delta_{ij} = \gamma_{jkl} (= \beta_{kli}), & i \neq j \neq k \neq l \neq 5 & (12).
\end{array}$$

Proof: Just apply (4.14), noting that here $d_2^2 = 45$, so $d_2 = 3\sqrt{5}$. $\square$

[3] this arrangement is given by the natural projective action of the icosahedral group on $\mathbb{P}^2$, and the six points are the five-fold points of the arrangement and form a single orbit

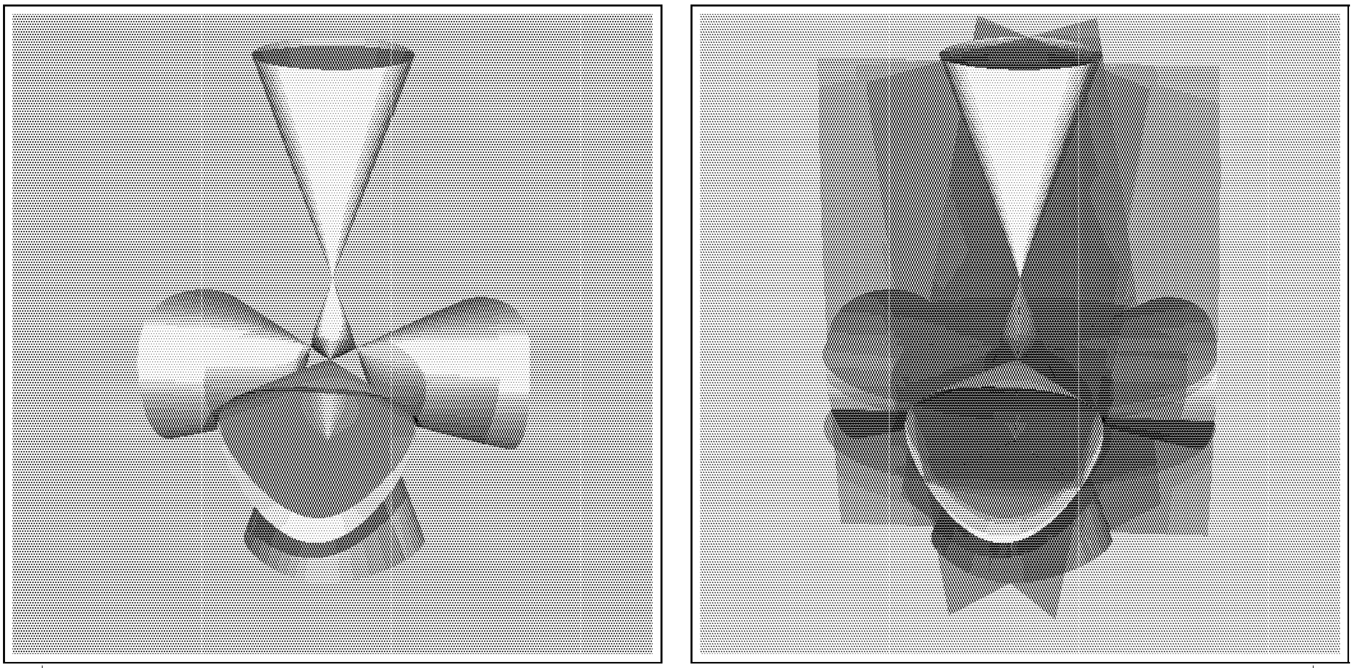
[4] via Cremona transformations of $\mathbb{P}^2$, described below, any such triple intersection in a tritangent on a cubic surface is equivalent to an intersection of three lines c_{ij}

Figure 4.5: The coordinate tetrahedron and Sylvester pentahedron for the Clebsch cubic.



The first picture shows the coordinate tetrahedron, which is given by the hyperplanes of (4.9). These are all Eckard tritangents, which is quite visible, at least for one of the planes. The second picture shows the corresponding Sylvester pentahedron of $\mathfrak{C}$, which in this case consists of tritangent planes.

The Hessian of $\mathfrak{C}$ and the Sylvester pentahedron



To get a nice equation for this cubic in homogenous coordinates on $\mathbb{P}^3$, we utilize the tetrahedral coordinates used above. This time we would like four of the ten Eckhard planes to comprise this coordinate tetrahedron, i.e., we set

$$u_0 := y_1 + y_5,\ u_1 := y_2 + y_5,\ u_2 = y_3 + y_5,\ u_3 := y_4 + y_5, \tag{4.9}$$

and since we have the linear relation $\sum y_i = 0$, the last coordinate can be expressed as $u_3 = -y_1 - y_2 - y_3$, and the sum of the u_i is $3y_5$, which is one of the Sylvester planes (they are the $y_i = 0$). To get an affine picture we this time set:

$$u_0 = x,\ u_1 = y,\ u_2 = z,\ u_3 = 1/4(1 - x - y - z),$$

i.e., this time we take the plane at infinity to be $u_0 + u_1 + u_2 + 4u_3 = 0$. Then, we must invert the equations (4.9), and insert these expressions into the equation (4.8). This gives:

$$\begin{array}{c} y_5 = \frac{1}{3}(u_0 + u_1 + u_2 + u_3), \quad y_1 = -\frac{1}{3}(-2u_0 + u_1 + u_2 + u_3), \\ y_2 = -\frac{1}{3}(u_0 - 2u_1 + u_2 + u_3), \quad y_3 = -\frac{1}{3}(u_0 + u_1 - 2u_2 + u_3), \\ y_4 = -\frac{1}{3}(u_0 + u_1 + u_2 - 2u_3), \end{array} \tag{4.10}$$

and this is inserted into (4.8), giving us the equation for $\mathfrak{C}$ in the u_i coordinates. With this, we can again derive the affine equation, and the result is pictured in Figure 4.4.

The equations for the lines in the u_i variables are convenient for the purpose of drawing them. We have

$$0 = y_i + y_j = y_k + y_l = -\frac{1}{3}(u_0 + u_1 + u_2 + u_3) + \frac{1}{2}(u_i + u_j) = -\frac{1}{3}(u_0 + u_1 + u_2 + u_3) + \frac{1}{2}(u_k + u_l);$$

$$0 = y_i + y_j = y_k + y_5 = -\frac{1}{3}(u_0 + u_1 + u_2 + u_3) + \frac{1}{2}(u_i + u_j) = u_k;$$

$$\left\{ \begin{array}{rcl} 0 & = & (-5 - 3\sqrt{5})(y_i + y_5) - (-5 + 3\sqrt{5})(y_i + y_5) = (-5 - 3\sqrt{5})u_i - (-5 + 3\sqrt{5})u_j \\ & = & (-5 - 3\sqrt{5})(y_j + y_5) - (1 + 3\sqrt{5})(y_k + y_l) = \\ & & (-5 - 3\sqrt{5})u_j - (1 + 3\sqrt{5})(-\frac{1}{3}(u_0 + u_1 + u_2 + u_3) + \frac{1}{2}(u_k + u_l)) \end{array} \right.$$

In Figure 4.5 we show both the coordinate tetrahedron of Eckard planes, as well as the Sylvester pentahedron for $\mathfrak{C}$. These are given, in affine coordinates, by

$$\begin{array}{c} y_1 = -\frac{1}{18}(-9x + 3y + 3z + 1), \quad y_2 = -\frac{1}{18}(3x - 9y + 3z + 1), \\ y_3 = -\frac{1}{18}(3x + 3y - 9z + 1), \quad y_4 = -\frac{1}{18}(3x + 3y + 3z - 1), \\ y_5 = \frac{1}{9}(3x + 3y + 3z + 1). \end{array}$$

Note that these are indeed tritangent planes (though not Eckhard planes).

4.1.4 Del Pezzo surfaces

The modern point of view is to consider cubic surfaces as del Pezzo surfaces of degree 6. The combinatorics of the 27 lines, as listed in Table 4.1, is then encoded in the Picard group of the del Pezzo surface. In fact, the complement in $\mathrm{Pic}(S)$ to the hyperplane section, call it $\mathrm{Pic}^0(S)$, is isomorphic to the root lattice of E_6. The equation of the surface is given by the embedding of S by means of the linear system of elliptic curves through the six given points.

Fix six points in $\mathbb{P}^2$, say $x = (p_1, \ldots, p_6)$, such that the p_i are in general position, i.e., no three lie on a line, and not all six lie on a conic. Let $\widehat{\mathbb{P}}^2_x$ denote the blow up of $\mathbb{P}^2$ at all six points, $\varrho_x : \widehat{\mathbb{P}}^2_x \longrightarrow \mathbb{P}^2$. Consider the following curves as classes in $\mathrm{Pic}(\widehat{\mathbb{P}}^2_x)$:

i) $a_1, \dots, a_6$, the exceptional divisors over $(p_1, \dots, p_6)$;

ii) $b_1, \dots, b_6$, b_i the proper transform of the conic q_i passing through all points $p_j,\ j \neq i$;

iii) c_{ik}, the proper transform of the line $\overline{p_i p_k}$.

If we consider the surface $\widehat{\mathbb{P}}^2_x$, we have $H^2(\widehat{\mathbb{P}}^2_x, \mathbb{Z}) = [l]\mathbb{Z} \oplus_i \mathbb{Z}a_i$. Let Q be the intersection form on $H^2(\widehat{\mathbb{P}}^2_x, \mathbb{Z})$; then the classes $a_i,\ b_i,\ c_{ij}$ fulfill $Q(a_i, a_i) = Q(b_i, b_i) = Q(c_{ij}, c_{ij}) = -1$. In a well-known manner one takes a rank 6 subset, which is isomorphic to the root lattice of type $\mathbf{E_6}$. For details see [Man] or [DO] and references therein. Consider the orthocomplement of the canonical class on $\widehat{\mathbb{P}}^2_x$, and denote this by $\mathrm{Pic}^0(\widehat{\mathbb{P}}^2_x)$. Recall that the anti-canonical class is $3l + \sum_{i=1}^6 a_i$, and that the anti-canonical embedding of $\widehat{\mathbb{P}}^2_x$ is as a cubic surface. Consequently we may view $\mathrm{Pic}^0(\widehat{\mathbb{P}}^2_x)$ as the orthocomplement of the hyperplane section class of $\mathrm{Pic}(S_x)$, where S_x is the cubic surface which is the anti-canonical embedding. The following elements λ with $Q(\lambda, \lambda) = -2$ form a basis of $\mathrm{Pic}^0(S_x)$:

$$\begin{aligned} \alpha_0 &= l - a_1 - a_2 - a_3 \\ \alpha_1 &= a_1 - a_2 \\ \alpha_2 &= a_2 - a_3 \\ \alpha_3 &= a_3 - a_4 \\ \alpha_4 &= a_4 - a_5 \\ \alpha_5 &= a_5 - a_6 \end{aligned} \tag{4.11}$$

These also form a base of a root system of type $\mathbf{E_6}$, by taking $\alpha_1, \dots, \alpha_5$ as the sub-root system of type $\mathbf{A_5}$. Since the classes $a_i,\ b_i,\ c_{ij}$ are exceptional, they all represent elements of $\mathrm{Pic}^0(S_x)$. This leads to the following exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z}^{24} \longrightarrow \mathbb{Z}^{45} \longrightarrow \mathbb{Z}^{27} \longrightarrow \mathrm{Pic}^0(S_x) \longrightarrow 0 \tag{4.12}$$

which was already discussed in the section on the Segre cubic.

4.1.5 Coble's hexahedral form

Taking the embedding $\varepsilon : \widehat{\mathbb{P}}^2_x \longrightarrow S_x \subset \mathbb{P}^3$ given by the linear system of cubic curves through the six points, one sees without difficulty that $a_i,\ b_i,\ c_{ij}$ are all lines on S_x, and that their intersections are given by the 36 double sixes (4.1), so we have reproduced the situation studied above. To give an explicit form of the equation of S_x, it seems that Coble's hexahedral form is most convenient. This is given by

Theorem 4.1.13 ([C1], see also [DO]) *Let $x = (p_1, \dots, p_6)$ be a set of six points in general position in $\mathbb{P}^2$, and let $\widehat{\mathbb{P}}^2_x$ be the corresponding del Pezzo surface of degree 6. There exist six cubics $(a, b, c, d, e, f) \in H^0(\mathbb{P}^2, \mathcal{O}(3))$ on the six points, such that there are coefficients $(\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f})$, satisfying the following conditions*

i) $a + \cdots + f = 0$;

ii) $\overline{a} + \cdots + \overline{f} = 0$;

iii) $\overline{a}a + \cdots + \overline{f}f = 0$,

and such that the equation defining the cubic surface $\varepsilon(\widehat{\mathbb{P}}^2_x)$ is

$$a^3 + \cdots + f^3 = 0.$$

We note that the equation iii) is just the equation of a hyperplane in the symmetric $\mathbb{P}^4$, and the equation for the cubic surface shows this is just a hyperplane section of the Segre cubic. Hence we have

Theorem 4.1.14 *Given $x = (p_1, \ldots, p_6)$ as in 4.1.13 and the coordinates $(a, \ldots, f)$, there exists a hyperplane $H_{(\overline{a},\ldots,\overline{f})} \in (\mathbb{P}^4)^\vee$, given by $H_{(\overline{a},\ldots,\overline{f})} = \{\overline{a}a + \cdots + \overline{f}f = 0\}$, such that*

$$S_x = \mathcal{S}_3 \cap H_{(\overline{a},\ldots,\overline{f})}$$

is a cubic surface isomorphic to $\varepsilon(\widehat{\mathbb{P}}^2_x)$. If one fixes an ordering of the six points $x = (p_1, \ldots, p_6)$, then any choice of double six on $\widehat{\mathbb{P}}^2_x$ gives rise to a marked cubic surface, and for this marking the choice of $(a, \ldots, f)$ is unique. □

So in the hexahedral form, a marked cubic surface is given by a hyperplane section of the Segre cubic. The exceptional locus (of such hyperplane sections) can be determined from this description: any hyperplane containing one of the 15 Segre planes (3.27) meets $\mathcal{S}_3$ in a *degenerate* cubic surface.

As already mentioned, Cremona had noticed that the equations of the lines and tritangents can be directly expressed for a cubic surface given in the hexahedral form of Theorem 4.1.14. In terms of the coordinates $a, \ldots, f$ used above, they can be given as follows.

i) The 15 Segre planes (3.27) on the Segre cubic intersect the hyperplane sections $H_{(\overline{a},\ldots,\overline{f})}$ in the 15 lines of the type
$$a + d = b + e = c + f = 0 \tag{4.13}$$
(which are lines on $S_x = \mathcal{S}_3 \cap H_{(\overline{a},\ldots,\overline{f})}$ if $\overline{a}a + \cdots + \overline{f}f = 0$); taking any two of these equations defines a tritangent plane.

ii) The 12 other lines can be given by
$$\left\{ \begin{array}{rcl} (be - d_2)(b+e) - (cf + d_2)(c+f) & = & 0 \\ (cf - d_2)(c+f) - (ad + d_2)(a+d) & = & 0 \\ (ad - d_2)(a+d) - (be + d_2)(b+e) & = & 0, \end{array} \right. \tag{4.14}$$
where $d_2^2 = \sigma_2(\overline{a}, \ldots, \overline{f})^2 - 4\sigma_4(\overline{a}, \ldots, \overline{f}) = 0$ is the condition on the six points that they lie on a conic in $\mathbb{P}^2$.

Any two equations of the left hand side of (4.14) define the remaining 30 tritangents, by permutations of the letters, such that for an odd permutation, the sign of d_2 in (4.14) is changed accordingly (see [C1], I, p. 173).

If one considers the six points in $\mathbb{P}^2$ up to permutation, then the invariant algebra is given by

Theorem 4.1.15 ([C1], I, p. 176) *The invariants $\sigma_2, \ldots, \sigma_6, d_2\sqrt{d}$ consistute a rational and integral complete system of (Σ_6-) invariants of $\mathbf{P}^2_6$ (see (3.85)), where*

$$d = \prod(\overline{a} - \overline{b})^2.$$

$d = 0$ is the condition that three lines c_{ij}, c_{jk}, c_{ik} meet at a point (i.e., determine an Eckard point on the cubic surface).

The actual symmetry group of the 27 lines – $W(E_6)$ – contains this Σ_6 (acting regularly on the cubics). It is extended to $W(E_6)$ by a *Cremona transformation* in the plane: the quadratic transformation given by blowing up three of the points a_4, a_5, a_6 and then blowing down the joining lines c_{45}, c_{46}, c_{56} yields the switch of double sixes:

$$N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{bmatrix} = N_{123},$$

which, together with Σ_6, generates $W(E_6)$. This gives a ***rational*** action of $W(E_6)$ on $\mathbf{P}_6^2$. Hence one can ask for the set of invariants under the $G_{6,2}$[5] action.

In [C1], III, Coble determines invariants for $G_{6,2}$, as expressions in the coordinates $\overline{a}, \ldots, \overline{f}$. These, in turn, may be expressed in terms of x, y, z, t, u, where the six points in $\mathbb{P}^2$ are brought into the form

$$\left(\begin{array}{cccc|cc} 1 & 0 & 0 & 1 & x & z \\ 0 & 1 & 0 & 1 & y & t \\ 0 & 0 & 1 & 1 & u & u \end{array} \right). \tag{4.15}$$

The basic idea is to apply the Clebsch transference principle (see section B.2.4) to the surface $S_{(\overline{a},\ldots,\overline{f})}$, which by the discussion above is a hyperplane section of the Segre cubic. From this it follows that the invariants of the quatenary cubic form (B.49) determine corresponding invariants of the hexahedral form as derived in section B.4.2 in the Appendix. Passing on to expressions in the (x, y, z, t, u), note that the $(\overline{a}, \ldots, \overline{f})$ are quadratic in the (x, y, z, t, u) (see [C1] I, p. 196).

$$\begin{array}{rclrcl} 3a &=& \rho - 3(ux + ut), & 3d &=& \rho - 3(uy + uz), \\ 3b &=& \rho - 3(ux + yz), & 3e &=& \rho - 3(uy + xt), \\ 3c &=& \rho - 3(ut + yz), & 3f &=& \rho - 3(uz + xt), \end{array} \tag{4.16}$$
$$\rho = u(x + y + z + t) + xt + yz.$$

Then we have

Theorem 4.1.16 *A complete system of invariants is given by:*

- *for the quatenary cubic: $i_8, i_{16}, i_{24}, i_{32}, i_{40}$ and i_{100} of degrees indicated and of weights 6, 12, 18, 24, 30 and 75 (see (B.49), where they are denoted by I_k instead of i_k).*
- *for a hexahedral cubic: $I_6, I_{12}, I_{18}, I_{24}, I_{30}$ and I_{75} of the degrees indicated in $(\overline{a}, \ldots, \overline{f})$ (see (B.56), (B.57) and (B.58) for these).*
- *for $G_{6,2}$: $j_{10}, j_{20}, j_{30}, j_{40}, j_{50}$ and j_{125} of the degrees indicated in (x, y, z, t, u).*

Of course these systems are related, but, and that is the whole point, in a very non-trivial manner. In this respect one defines the following related problems.

Definition 4.1.17 The ***equation problem*** for $G_{6,2}$ is: given numerical values of $j_{10}, \ldots, j_{125}$, calculate the coordinates (x, y, z, t, u) of a point $P \in \mathbf{P}_6^2$ for which the j_k take the assigned values.

The next problem concerns the subgroup $\Gamma_{6,2} \subset G_{6,2}$. Since (semi-)invariants of $G_{6,2}$ which change sign under odd permutations are genuine invariants of $\Gamma_{6,2}$, the invariants of the latter can be deduced from those of the former (cf. section B.1.2.1). As Coble shows, the invariant j_{40} may be taken to be the ***discriminant*** Δ of the cubic surface (in fact, i_{32}, I_{24} and j_{40} may ***all*** be represented by the discriminant, each expressed in the appropriate variables), and $\sqrt{\Delta}$ can be taken as an invariant for $\Gamma_{6,2}$. To see how this occurs, note that the discriminant has rational factors corresponding to the

[5] we are using Coble's notation for the Cremona groups; as abstract groups $G_{6,2} \cong W(E_6)$ and $\Gamma_{6,2} \cong G_{25,920}$

conditions (on the six points in the plane) that the so defined cubic surface is ***singular***: either all six points lie on a conic, three lie on a line, or two points coincide in some direction. Hence, we have "$\Delta = \delta \prod \delta_{ijk}$", where δ_{ijk} expresses the condition that three points (labeled p_i, p_j, p_k) lie on a line. This expression, however, is not correct, as in order to express Δ rationally in terms of $\overline{a}, \ldots, \overline{f}$, we must utilize the relation (here $\delta = d_2$, where d_2 was defined above):

$$\delta^2 = \sigma_2^2 - 4\sigma_4,$$

which means the discriminant will divide δ^2, hence by symmetry, will also divide δ_{ijk}^2, so what we actually have is

$$\Delta = \delta^2 \prod \delta_{ijk}^2.$$

This expression is now of degree 24 in $\overline{a}, \ldots, \overline{f}$, but in these variables it is a square, and up to sign it makes sense to speak of $\sqrt{\Delta} = \delta \prod \delta_{ijk}$. Note that

$$\delta_{123}\delta_{456} = (\overline{a} + \overline{b} + \overline{c}) = -(\overline{d} + \overline{e} + \overline{f}),$$

etc. A definite sign can be given to $\sqrt{\Delta}$ by the assignment

$$\sqrt{\Delta} = d_2 \prod_{i,j \in \{\overline{b}, \ldots, \overline{f}\}} (\overline{a} + i + j), \tag{4.17}$$

and d_2 is given the sign

$$d_2 = \begin{vmatrix} (341)(561) & (531)(461) \\ (342)(562) & (532)(462) \end{vmatrix}, \tag{4.18}$$

where $(ijk) =$ the 3×3 minor of (4.15) given by the $(i, j, k)^{th}$ columns. Then

Lemma 4.1.18 *A complete system of invariants for* $\Gamma_{6,2}$ *is given by*

$$j_{10}, j_{20}, j_{30}, \sqrt{\Delta}, j_{50} \textit{ and } j_{125}.$$

The skew invariant i_{100} or I_{75} or j_{125} has the same behavior under $G_{6,2}$ and $\Gamma_{6,2}$ as $\sqrt{\Delta}$: it is invariant under $\Gamma_{6,2}$ and changes sign under all odd elements of $G_{6,2}$. The invariant is a product over 45 factors, each expressing the condition that a given tritangent plane contains an Eckard point.

Definition 4.1.19 The *form problem* for $\Gamma_{6,2}$ is: given numerical values for $j_{10}, j_{20}, j_{30}, \sqrt{\Delta}, j_{50}$, calculate the ratios of the coordinates (x, y, z, t, u) of a point $p \in \mathbf{P}_6^2$ for which the j_k take the assigned values.

Finally, it is important to know whether the three sets of invariants of Theorem 4.1.16 are ***covariantly*** equivalent. This means that they change in the same way upon affecting some permutation of the 27 lines. For the invariants I_k this can be achieved by using the Clebsch transference principle, since this preserves the covariant relations, and is described in the appendix.

4.2 Solving algebraic equations by means of theta functions

In this section we review the general procedure, developed in the last century, for solving algebraic equations by means of transcendental functions, in particular theta functions, presenting explicitly the example of the general quintic equation, the Galois group of which is Σ_5. Part of the reason for presenting this "old" theory in such detail is because it forms, in some sense, the beginnings of the theory of arithmetic quotients and their geometry. This will be much extended in the example of the equation for the 27 lines of a general cubic surface, where much more geometry comes into play. But still much is based on this case, which then, in addition to being part of the history of our subject, serves also as a pattern for a more interesting example.

4.2.1 Algebraic equations

To start we recall a few general facts about the roots of an algebraic equation. These relate the solution of a given equation to solutions of other equations, so-called resolvents. As a first step, there is a general method to simplify a general degree n equation

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0. \tag{4.19}$$

It follows from Galois theory that (4.19) can be solved by means of ***radicals***, if and only if the Galois group is ***solvable***. Hence, supposing the Galois group to be simple, this method does not yield a solution. Letting $(\alpha_1, \ldots, \alpha_n)$ denote the roots of (4.19), the Galois group of the equation (4.19) is the Galois group of the field extension

$$K = k(\alpha_1, \ldots, \alpha_n), \tag{4.20}$$

where the coefficients a_i of (4.19) are assumed to lie in k. If the equation is general, then the Galois group is Σ_n. (It is not so easy to explicitly write down a "general" equation; see for example Lang's book "*Algebra*" for examples of general quintics (a criterion due to Artin)). Let Δ denote the discriminant of (4.19); then the Galois group $Gal(K|k(\sqrt{\Delta})) \cong A_n$ is simple.

Now let $\psi \in k[X]$ be a polynomial of degree $\leq n-1$, and set

$$Q(Y) = \tau_\psi(P)(Y) := \prod (Y - \psi(\alpha_i)) \in k[Y]. \tag{4.21}$$

The polynomial $\tau_\psi(P)$ is called the ***Tschirnhaus transformation*** of P by ψ. Assume that P and Q have no multiple roots (which of course holds if they are general). Then we have

Lemma 4.2.1 *The roots of P can be rationally calculated in terms of the roots of Q.*

Proof: Let $\beta_1, \ldots, \beta_n$ be the roots of Q, $\beta_i = \psi(\alpha_i)$. Set

$$\lambda(Y) = Q(Y) \cdot \left(\frac{\alpha_1}{Y - \beta_1} + \cdots + \frac{\alpha_n}{Y - \beta_n} \right).$$

$\lambda(Y)$ can be written $\lambda(Y) = \sum_\mu \prod_{i \neq \mu} (Y - \beta_i)\alpha_\mu$, and the coefficients, being rationally expressible in terms of the elementary symmetric functions of the roots, are in k. Hence $\lambda(\beta_k) = \prod_{i \neq k}(\beta_k - \beta_i)\alpha_k$, and consequently

$$\alpha_k = \frac{\lambda(\beta_k)}{\prod_{i \neq k}(\beta_k - \beta_i)} = \frac{\lambda(\beta_k)}{Q'(\beta_k)}, \tag{4.22}$$

giving the desired expression. □

4.2.2 Resolvents

Similarly, let $y \in K$ be given, and let $y_1 = y, y_2, \ldots, y_m$ be the conjugates of y under the action of the Galois group $G = Gal(K|k)$. Then

$$R(X) := \prod_{i=1}^{m} (X - y_i) \tag{4.23}$$

is a polynomial with coefficients in k. It is called a ***resolvent*** of $P(X)$. In the particular case that the stabilizer of y_i in G is the identity element, we have $m = |G|$ and R is called a ***Galois resolvent***. As above, set

$$\lambda(X) = R(X) \left(\frac{\sigma_1(\alpha_1)}{X - y_1} + \cdots + \frac{\sigma_m(\alpha_1)}{X - y_m} \right),$$

where $y_i = \sigma_i(y_1)$ for $\sigma_i \in G$. Then the same argument as above shows

$$\sigma_k(\alpha_1) =: \alpha_k = \frac{\lambda(y_k)}{R'(y_k)},$$

so again

Lemma 4.2.2 *Let $R(X)$ be a resolvent of $P(X)$. Then the roots of $P(X)$ can be rationally calculated in terms of the roots of $R(X)$.*

Theorem 4.2.3 (Reduction of equations, Tschirnhaus 1683) *By a linear tranformation, or by a quadratic Tschirnhaus transformation whose coefficients involve a* single square root*, any equation*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

can be reduced to a principal *equation*

$$x^n + c_{n-3}x^{n-3} + \cdots + c_1 x + c_0 = 0.$$

The square root which must be introduce above is called an *accessory irrationality*, as it is only introduced to put the given equation is some normal form. Going one step further,

Theorem 4.2.4 *By adjunction of a cube root and three square roots any equation of degree n can be reduced to the form*

$$x^n + c_{n-4}x^{n-4} + \cdots + c_1 x + c_0 = 0.$$

For $n = 5$ this is due to Bring (1786), and the Bring form of a quintic equation is (cf. (B.25))

$$y^5 + 5by + c = 0. \tag{4.24}$$

The general method of solving algebraic equations amounts to finding resolvents of a given equation, which one knows how to solve for some reason or another. For example, a "pure" equation is one which can be reduced to the form $x^n = a$, which can then be solved by $x = \exp(\frac{1}{n}\ln a)$. Many equations of interest can be reduced to equations involving theta functions, of which the modular equation, to be discussed next, is the prototype.

4.2.3 The modular equation

Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, given in the Weierstraß form

$$y^2 = 4x^3 - g_2 x - g_3, \tag{4.25}$$

with $g_2 = 60 \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \frac{1}{\omega^4}$, $g_3 = 140 \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \frac{1}{\omega^6}$ and discriminant $\Delta = g_2^3 - 27g_3^2$. The J-invariant is $J = g_2^3/\Delta$, and as Λ varies, J becomes a modular function (for $PSL(2,\mathbb{Z})$) on the upper half plane $\mathbb{S}_1 = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$ (where $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$). The *modular equation* is: given $u \in \mathbb{C}$, find $\tau \in \mathbb{S}_1$, such that

$$J(\tau) = u. \tag{4.26}$$

A solution τ of (4.26) can be constructed as follows. Given $u \in \mathbb{C}$,

$$E_u : \quad y^2 = 4x^3 - \frac{27u}{u-1}(x+1) \tag{4.27}$$

is the Weierstraß equation of an elliptic curve E_u with J-invariant equal to u. To find $\tau \in \mathbb{S}_1$ with $E_u = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, one calculates the ***periods*** of E_u,

$$\omega_i = \int_{\gamma_i} \frac{dx}{y} = \int_{\gamma_i} \frac{dx}{\sqrt{4x^3 - \frac{27u}{u-1}(x+1)}}, \tag{4.28}$$

where (γ_1, γ_2) is a $\mathbb{Z}$-basis of $H_1(E_u, \mathbb{Z})$. Then the ***solution*** of (4.26) is

$$\tau_u = \omega_1/\omega_2; \quad J(\tau_u) = u. \tag{4.29}$$

Geometrically, solving (4.26) is equivalent to finding an inverse image of u under

$$J : \mathbb{S}_1 \longrightarrow \Gamma\backslash\mathbb{S}_1, \quad \Gamma = SL(2, \mathbb{Z}).$$

A solution $\tau_u \in \mathbb{S}_1$ is determined up to the action of Γ, corresponding to the choice of a basis (γ_1, γ_2) in (4.28).

The division problem

Consider the two elliptic curves E_τ and $E_{n\tau}$. Since the fundamental parallelogram of $E_{n\tau}$ consists of n copies of the fundamental paralellogram of E_τ, one sees that there is a map $E_\tau \xrightarrow{\phi_n} E_{n\tau}$ given by the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "\Lambda_{n\tau}"] \arrow[dr, "\Lambda_\tau"'] & & E_{n\tau} \\ & E_\tau \arrow[ur, "\phi_n"'] & \end{tikzcd} \tag{4.30}$$

and since $[\Lambda_\tau : \Lambda_{n\tau}] = n$, ϕ_n is a finite map of degree n. The kernel is finite and consists of points $k\tau$, $1 \leq k \leq n$. A surjective morphism

$$\phi : E \longrightarrow E'$$

of elliptic curves is called an ***isogeny***, if $|\mathrm{Ker}(\phi)| < \infty$.

Division problem: Given an isogeny $\phi : E_\tau \longrightarrow E'_{\tau'}$, determine the relation between the values $\xi(\tau)$ and $\xi(\tau')$ for any modular form ξ.

Theorem 4.2.5 *Given an isogeny $\phi : E \longrightarrow E'$, there are algebraic relations between the elliptic functions on E and on E'.*

Proof: Let $K(E)$ and $K(E')$ be the function fields; each is generated by the elliptic functions. Then from the isogeny we get that $K(E)$ is an algebraic extension of the field $K(E')$, and this implies that there are algebraic relations between the elliptic functions on E and E'. $\square$

Corollary 4.2.6 *The same holds for invariants of $K(E)$ (for example g_2, g_3, Δ, etc.).*

In fact, for the isogenies ϕ_n the extension of function fields is even Galois, with Galois group $\mathbb{Z}/n\mathbb{Z}$.

With isogenous curves one gets other equations related to the modular equation (4.26). For this, consider the principal congruence subgroup $\Gamma(N) \subset SL(2, \mathbb{Z})$ and the corresponding factorization

$$\begin{tikzcd} \mathbb{S}_1 \arrow[rr, "J"] \arrow[dr, "J_N"'] & & \Gamma\backslash\mathbb{S}_1 \\ & \Gamma(N)\backslash\mathbb{S}_1 \arrow[ur, "\pi_N"'] & \end{tikzcd} \tag{4.31}$$

The function J_N is called the level N modular function. Note that since $\Gamma(N)$ is a normal subgroup of $SL(2,\mathbb{Z})$, the finite cover

$$\Gamma(N)\backslash\mathbb{S}_1 \xrightarrow{\pi_N} \Gamma\backslash\mathbb{S}_1 \tag{4.32}$$

is a Galois cover, with Galois group $P\Gamma/P\Gamma(N)$. In particular, for $N=5$, we have the identification

$$P\Gamma/P\Gamma(5) \;\cong\; PSL(2,\mathbb{Z}/5\mathbb{Z}) \;\cong\; A_5, \tag{4.33}$$

the alternating group on five letters, which is the Galois group $Gal(K|k(\sqrt{\Delta}))$ of a general equation of degree 5, i.e., for K as in (4.20) with $n=5$.

The solutions to these modular equations can be found somewhat more directly than with periods as above in terms of hypergeometric functions. These functions arise as solutions of the Picard-Fuchs equations for the corresponding family of elliptic curves with level N structures, that is pairs (E_u, L), where E_u is an isomorphism class of elliptic curves as in (4.27) and L is a level N structure on E_u. These Picard-Fuchs equations for $N=2,3,4$ or 5 take the form (cf. [Kl3], (38), p. 80):

$$y'' + \frac{y'}{u} + \frac{y}{4(u-1)^2u^2}\cdot\left\{-\frac{1}{\nu_2^2} + u\left(\frac{1}{\nu_2^2}+\frac{1}{\nu_3^2}-\frac{1}{\nu_1^2}+1\right)-\frac{u^2}{\nu_3^2}\right\} = 0, \tag{4.34}$$

for triples $(\nu_1,\nu_2,\nu_3) = (2,2,3),(2,3,3),(2,3,4),(2,3,5)$ for $N=2,3,4$ and 5, respectively. Solutions to the modular equations can be given in terms of Riemann P-functions ([Kl3], p. 81)

$$P\begin{pmatrix} \frac{1}{2\nu_2} & \frac{1}{2\nu_3} & \frac{1}{4} & \\ -\frac{1}{2\nu_2} & -\frac{1}{2\nu_3} & \frac{3}{4} & ,u \end{pmatrix}. \tag{4.35}$$

The only singularities of (4.34) occur at $u = J(\tau) = 0, 1, \infty$, corresponding to finite torsion in $SL(2,\mathbb{Z})$ at i, $\varrho = e^{2\pi i/3}$ and the parabolic at ∞. In this way, the periods (4.28) can be calculated in terms of hypergeometric functions.

Jacobi's equation (1829)

The following lines are taken directly from Jacobi's "*Fundamenta nova theoriae functionum ellipticarum*", which appeared in 1829.

Let

$$K := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}},\quad K' := \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-(\kappa')^2\sin^2\phi}},\quad \kappa^2+(\kappa')^2 = 1.$$

If

$$\Lambda := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}},\quad \Lambda' := \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-(\lambda')^2\sin^2\phi}},\quad \lambda^2+(\lambda')^2 = 1$$

and

$$\begin{pmatrix}\Lambda\\ \Lambda'\end{pmatrix} = A\begin{pmatrix}K\\ K'\end{pmatrix} \quad \text{with } A = \begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad ad-bc = n,$$

then, setting $u = \sqrt[4]{\kappa}$, $v = \sqrt[4]{\lambda}$ we get for prime n equations of degree $(n+1)$ between u and v, for example

$$\begin{array}{ll} n=3: & u^4 - v^4 - 2uv(1-u^2v^2) = 0\\ n=5: & u^6 = v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-u^4v^4). \end{array} \tag{4.36}$$

The latter equation is "Jacobi's 6^{th} degree equation for the transformations of order five of the elliptic functions." Putting the above into more modern language we have: K, K' are nothing but (up to a factor) the periods ω_1 and ω_2, and Λ, Λ' are the periods corresponding to the elliptic curve $E_{n\tau}$ (see (4.30)). The functions κ and λ are rational functions of $J(\tau)$ and $J(n\tau)$:

$$J(\tau) = 1728 \cdot 16 \frac{(\kappa^4 + 14\kappa^2 + 1)^3}{\kappa^2(\kappa^2 - 1)^4}, \quad J(n\tau) = \text{ same with } \kappa \text{ replaced by } \lambda. \tag{4.37}$$

The function κ (resp. λ) corresponds to the Legendre normal form:

$$y^2 = (1 - x^2)(1 - \kappa^2 x^2) \quad (\text{resp. } y^2 = (1 - x^2)(1 - \lambda^2 x^2)) \tag{4.38}$$

for which all branch points of the double cover $E_\tau \longrightarrow \mathbb{P}^1$ (resp. for which all branch points of $E_{n\tau} \longrightarrow \mathbb{P}^1$) are finite (as opposed with the Weierstraß form, which has one of the branch points at infinity). If $J_1, \ldots, J_{s_n}$ denotes the J-invariants of all curves n-isogenous to E_τ, then these values satisfy a modular equation

$$\Phi_n(x, J) = 0 \tag{4.39}$$

with $\Phi_n \in \mathbb{Z}[X, Y]$, and ($n$ a prime) Φ_n has degree $(n+1)$ in both X and Y. For more details on the following, see [L], Chapter 5.

Lemma 4.2.7 *For general E and prime n, the Galois group of $\Phi_n(x, J)$ over $\mathbb{Q}(\varepsilon, J)$ (ε an n^{th} root of unity) is isomorphic to $PSL(2, \mathbb{Z}/n\mathbb{Z})$ acting on the $(n+1)$ roots of Φ_n in the J variable, $J_0, \ldots, J_n$ as on the points of $\mathbb{P}^1(\mathbb{F}_n)$.*

The equation for $n = 5$, when J is replaced by κ by means of (4.37), is nothing but Jacobi's equation of the sixth degree.

The important fact about the equation (4.36) is, by considering u as a ***parameter***, it is an equation of 6^{th} degree (in the variable v) with one parameter, ***which can be solved by means of elliptic functions.***

4.2.4 Hermite, Kronecker and Brioschi

In a paper "Sur la résolution de l'équation du cinquième degré" published in 1858, Hermite found the resolvent of fifth degree which derives from Jacobi's equation of sixth degree. This arises, because, as just mentioned, the modular equation has Galois group A_5, so there exists a resolvent of fifth degree (there exists a resolvent of any degree which divides 60), corresponding to a root with stabilizer A_4 which is of index five in A_5. He did this as follows: set $y = (v_\infty - v_0)(v_1 - v_4)(v_2 - v_3)$, where v_i are the ***roots*** of Jacobi's equation. The corresponding resolvent (4.23) is

$$y^5 - 2^4 \cdot 5^3 u^4 (1 - u^8)^2 \cdot y - 2^6 \sqrt{5^5} \cdot u^3 (1 - u^8)^2 (1 + u^8) = 0. \tag{4.40}$$

This takes the form of a "Bring equation" $t^5 - t - A = 0$ (see (4.24) and (B.25)) by setting

$$\begin{aligned} y &= 2\sqrt[4]{5^3} u \sqrt{1 - u^8}\, t \\ A &= \frac{2}{\sqrt[4]{5^5}} \cdot \frac{1 + u^8}{u^2 \sqrt{1 - u^8}}. \end{aligned} \tag{4.41}$$

Hence, since we have seen by a Tschirnhaus transformation any quintic equation can be brought into the Bring form (by adjoining a cube root and three square roots, see Theorem 4.2.4), which can be solved by the elliptic functions for the transformations of fifth order, this gives a solution of the quintic equation. A flow chart for this solution is given in Figure 4.6.

In the mentioned work of Jacobi, he applied Tschirnhaus transformations to the equation of sixth degree (4.36) (respectively of $(n+1)^{st}$ degree) to get other equations of sixth (resp. $(n+1)^{st}$) degree. These had the following amazing property:

Figure 4.6: Hermite's solution

The general quintic equation $x^5 + \cdots + a_1 x + a_0 = 0.$	Tschirnhaus transformation $\longrightarrow$ Introduces accessory irrationalities: one cube and three square roots	Bring's form of the quintic equation: $t^5 - t - A = 0.$
		$\downarrow$ Resolvent
$u = \sqrt{2}q^{1/8} \cdot \dfrac{\sum q^{2m^2+m}}{\sum q^{m^2}} \Rightarrow$ roots $v_\infty, \ldots, v_n$, given by replacing $q^{1/8}$ by: $q^{n/8},\ q^{1/8n}, \alpha q^{1/8n}, \ldots, \alpha^{n-1} q^{1/8n},\quad \alpha = e^{2\pi i/n}.$	solves $\longrightarrow$	Jacobi's modular equation of degree 6 for transformations of 5^{th} order of elliptic functions.

Theorem 4.2.8 (Jacobi, 1829) *The given set of equations satisfy: the square roots of the roots* $z_\infty, z_0, \ldots, z_{n-1}$ *can be* linearly *combined from* $\frac{n+1}{2}$ *elements* $\mathbf{A}_0, \ldots, \mathbf{A}_{\frac{n-1}{2}}$*:*

$$\begin{cases} \sqrt{z_\infty} = \sqrt{(-1)^{\frac{n-1}{2}} n \mathbf{A}_0} \\ \sqrt{z_\nu} = \mathbf{A}_0 + \varepsilon_\nu \mathbf{A}_1 + \cdots + \varepsilon^{\left(\frac{n-1}{2}\right)^2} \mathbf{A}_{\frac{n-1}{2}}, \end{cases}$$

where $\nu = 1, \ldots, n-1,\ \varepsilon = e^{\frac{2\pi i}{n}}$*, and there are choices of the square roots so that the given relations hold.*

Brioschi then showed how to form a "general" Jacobi equation of sixth degree: there are homogenous polynomials A, B, C of the $\mathbf{A}_i$ of degrees 2,6 and 10, such that

$$(z-A)^6 - 4A(z-A)^5 + 10B(z-A) + (5B^2 - AC) = 0 \tag{4.42}$$

is the most general form for a sextic such that the (square roots of the) roots can be linearly expressed in terms of three elements. Furthermore, he finds the resolvent of fifth degree for this equation:

$$x^5 + 10Bx^3 + 5(9B^2 - AC)B^2 x - \sqrt[4]{\frac{\Pi}{5^5}} = 0, \tag{4.43}$$

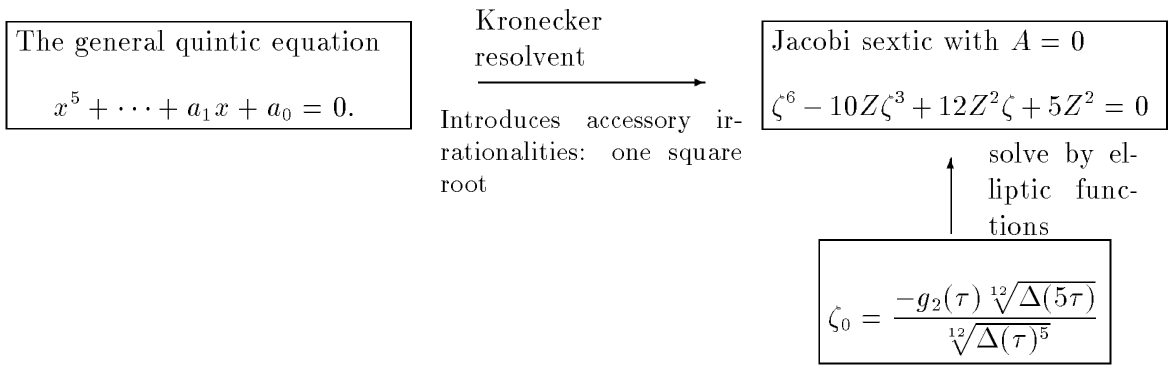
where Π is the discriminant of the equation (4.42). For $B = 0$ this is a Bring equation, and for $A = 0$, of the form

$$x^5 + 10Bx^3 + 45B^2 x - \sqrt[4]{\frac{\Pi}{5^5}} = 0. \tag{4.44}$$

Kronecker found that the sextic (4.42) with $A = 0$ could be solved by means of elliptic functions:

$$\zeta^6 - 10Z\zeta^3 + 12Z^2\zeta + 5Z^2 = 0 \tag{4.45}$$

Figure 4.7: Kronecker's solution



has a root

$$\zeta_0 = \frac{-g_2(\tau)\sqrt[12]{\Delta(5\tau)}}{\sqrt[12]{\Delta(\tau)^5}} \tag{4.46}$$

where

$$Z = J(\tau). \tag{4.47}$$

Furthermore he proved (most of)

Theorem 4.2.9 *A general quintic equation (after adjoining the square root of the determinant and hence reducing the Galois group to A_5) gives rise to a* sextic *resolvent which* is *a Jacobi equation (and hence can be solved by means of elliptic functions). By adjoining an additional square root it can be transformed into the form $A = 0$ above, but without such an irrationality cannot be reduced to a one-parameter equation.*

Kronecker claimed but did not prove the statement on the necessity of adjoining an accessory irrationality. Klein was to give a proof of this later. We sketch Kronecker's solution in the flow chart given in Figure 4.7.

4.2.5 The geometric description

Let

$$P(x) = x^5 + a_4x^4 + \cdots + a_1x + a_0 = 0 \tag{4.48}$$

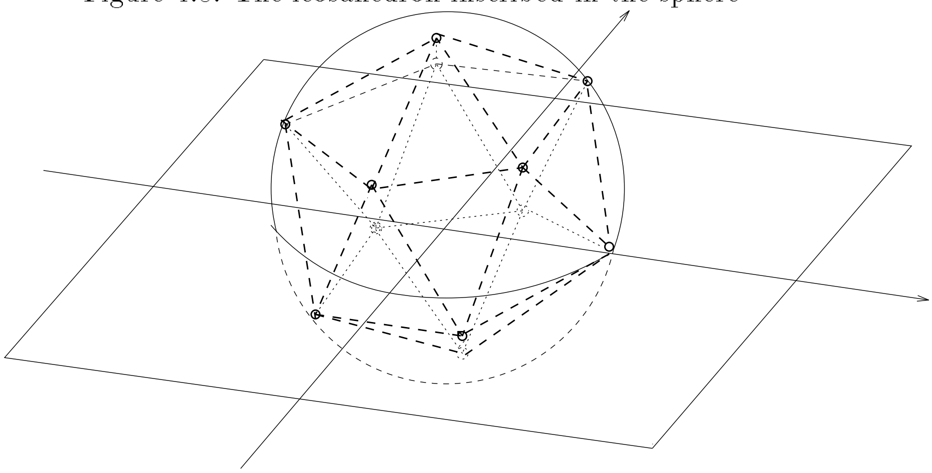
be a general quintic equation and let $x_1, \ldots, x_5$ denote its ***roots***. Then one can view $P(x)$ as determining a point

$$P(x): \quad (x_1 : \ldots : x_5) \in \mathbb{P}^4; \tag{4.49}$$

but since different ordering of the roots $x_1, \ldots, x_5$ give rise to the same polynomial $P(x)$, one can view the space of quintic polynomials as $\mathbb{P}^4/\Sigma_5$, where Σ_5 acts on $\mathbb{P}^4$ by permuting coordinates. Recall that the coefficients of the equation (4.48) are ($\pm$) the elementary symmetric functions of the roots. Hence, the Tschirnhaus transformation which transforms (4.48) into a quintic with $a_4 = 0$ corresponds to restricting attention to polynomials for which $\sum x_i = 0$. Since this is symmetric in the x_i, it follows that we have a well defined action of Σ_5 on the $\mathbb{P}^3$ given by that relation,

$$\mathbb{P}^3 = \{(x_1 : \ldots : x_5) \in \mathbb{P}^4 \Big| \sum x_i = 0\}.$$

Figure 4.8: The icosahedron inscribed in the sphere



Now we can describe the different Tschirnhaus transformations in terms of certain Σ_5-***invariant surfaces or curves*** in this $\mathbb{P}^3$. For example the "Hauptgleichung" of Felix Klein is

$$y^5 + \alpha y^2 + \beta y + \gamma = 0; \tag{4.50}$$

it corresponds to $\sum x_i^2 = 0$, i.e., to the quadric surface

$$Q = \{(x_1 : \ldots : x_5) \in \mathbb{P}^4 \Big| \sum x_i = \sum x_i^2 = 0\} \subset \mathbb{P}^3. \tag{4.51}$$

The "quintic resolvent with $A = 0$" (4.43) corresponds to $\sum x_i^3 = 0$, i.e., to the Clebsch diagonal cubic surface (see (4.8))

$$\mathfrak{C} = \{(x_1 : \ldots : x_5) \in \mathbb{P}^4 \Big| \sum x_i = \sum x_i^3 = 0\} \subset \mathbb{P}^3. \tag{4.52}$$

Finally the "Bring equation" (4.40) corresponds to $\sum x_i^2 = \sum x_i^3 = 0$, i.e. to the Bring curve

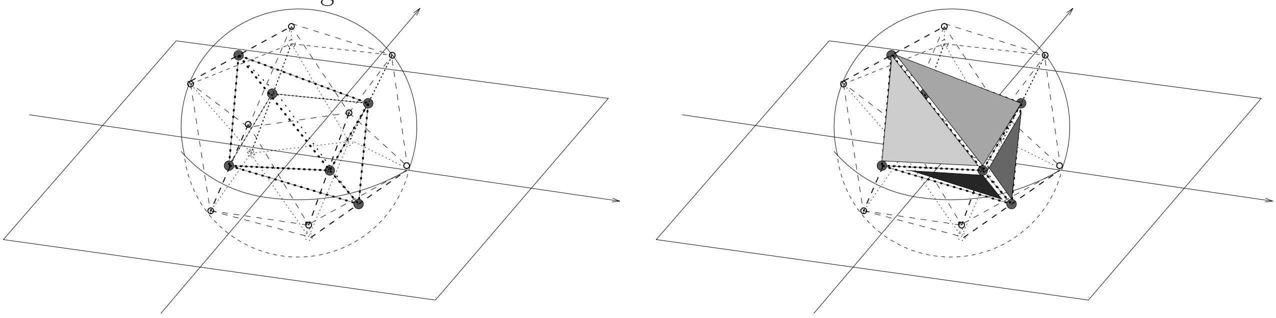
$$B = \{(x_1 : \ldots : x_5) \in \mathbb{P}^4 \Big| \sum x_i = \sum x_i^2 = \sum x_i^3 = 0\} \subset \mathbb{P}^3. \tag{4.53}$$

All of the above are invariant under the action of Σ_5. Recall also that the first two cases require an accessory square root, while in the third case one must adjoin a cube root and three square roots (see Theorem 4.2.3).

4.2.6 The icosahedron

There is a universal Galois resolvent for general quintic equations: the icosahedral equation. In studying this resolvent one can apply the geometry of the icosahedron, making the algebra very geometric. The icosahedral equation is easily solved in terms of transcendental functions, giving a "universal solution" to the problem of finding roots of quintic equations. Although studied by many authors as described above, it was Felix Klein who built the entire theory of quintic equations on these foundations, laid down in the classic [Kl3]. In the new edition, a modern mathematical translation is given by P. Slodowy, also explaining at which places assumptions must be made for the given statements to be valid. Also, the solution given by Klein is summarized in modern-day language, so there is no need to repeat it here. We just sketch the geometry and method of solution in as far as it pertains to our purpose: to study the equation of degree 27 for the lines on a cubic surface.

Figure 4.9: An inscribed octahedron in the icosahedron



The icosahedron can be inscribed in the Riemannian sphere, with verticies given by (see Figure 4.8):

$$z = 0, \infty, \varepsilon^{\nu}(\varepsilon + \varepsilon^4), \varepsilon^{\nu}(\varepsilon^2 + \varepsilon^3), \quad \varepsilon = e^{2\pi i/5}. \tag{4.54}$$

There are 12 vertices, 30 edges and 20 faces. As each symmetry of the icosahedron yields a linear transformation of $\mathbb{P}^1$, the icosahedral group $\mathcal{I}$ is naturally a (finite) subgroup of $PGL(2, \mathbb{C})$. But in fact, since the vertices are defined over the field $\mathbb{Q}(\varepsilon)$,

$$\mathcal{I} \subset PSL(2, \mathbb{Q}(\varepsilon)). \tag{4.55}$$

One can inscribe five octahedra in the icosahedron I: the 30 edges fall into five sets of six edges, the midpoints of which are the vertices of the octahedron. One such set is pictured in Figure 4.9.

It is easy to see that each symmetry of I affects an *even* permutation of the five octahedra, and hence

Lemma 4.2.10 $\mathcal{I} \cong A_5$, *the alternating group on 5 letters.*

Since $\mathcal{I}$ is a finite group of automorphisms of $\mathbb{P}^1$, one can find a finite set of generators of the ring of invariant functions. Clearly, if v_i denote the 12 vertices, w_j the 30 midpoints of the edges and u_k the 20 centerpoints of the faces, then we have invariants of degrees 12, 20 and 30:

$$\begin{aligned} f(z_0 : z_1) &= \prod_{i=1}^{12} (z_0(v_i)_1 - z_1(v_i)_0) = z_0 z_1 (z_0^{10} + 11 z_0^5 z_1^5 - z_1^{10}), \\ H(z_0 : z_1) &= \prod_{k=1}^{20} (z_0(u_k)_1 - z_1(u_k)_0) = \mathbf{Hess}(f) = \left| \frac{\partial^2 f}{\partial z_i \partial z_j} \right| \\ &= -(z_0^{20} + z_1^{20}) + 228(z_0^{15} z_1^5 - z_0^5 z_1^{15}) - 494 z_0^{10} z_1^{10}, \\ T(z_0 : z_1) &= \prod_{j=1}^{30} (z_0(w_j)_1 - z_1(w_j)_0) = \mathfrak{J}(f, H) \\ &= (z_0^{30} + z_1^{30}) + 522(z_0^{25} z_1^5 - z_0^5 z_1^{25}) - 10005(z_0^{20} z_1^{10} + z_0^{10} z_1^{20}). \end{aligned} \tag{4.56}$$

Here we have used the Hessian variety and Jacobian, see Definitions B.1.8 and B.1.7, respectively. Since there are only two coordinates, there is a relation between these forms, which is:

$$T^2 = -H^3 + 1728 f^5. \tag{4.57}$$

As a subgroup of $PGL(\mathbb{C})$, $\mathcal{I}$ acts on $\mathbb{P}^1$; let

$$q : \mathbb{P}^1 \longrightarrow \mathbb{P}^1/\mathcal{I} \cong \mathbb{P}^1 \tag{4.58}$$

denote the projection onto the quotient. An easy argument shows that

$$q(z_0 : z_1) = \frac{H^3(z_0 : z_1)}{1728 f^5(z_0 : z_1)}, \tag{4.59}$$

where f, H are the degree 12 and degree 20 invariants above. Let $u \in \mathbb{P}^1$ be an affine coordinate on the quotient; then the degree of q is 60, with the following branching at the points $u = 0, 1, \infty$:

u	$q^{-1}(u)$	$\lvert stab(z)\rvert, z \in q^{-1}(u)$
0	20 points	$\mathbb{Z}/3\mathbb{Z}$
1	30 points	$\mathbb{Z}/2\mathbb{Z}$
∞	12 points	$\mathbb{Z}/5\mathbb{Z}$

By Lemma 4.2.10 and (4.33) we have

Lemma 4.2.11 $\mathcal{I} \cong PSL(2, \mathbb{Z}/5\mathbb{Z}) \cong A_5$.

Moreover, the ***natural*** action of $PSL(2, \mathbb{Z}/5\mathbb{Z})$ on $\mathbb{P}^1$, with quotient morphism π_5 as in (4.32), can be identified with the ***natural*** action of $\mathcal{I}$ induced by the symmetries of the icosahedron:

Proposition 4.2.12 *The cover $\mathbb{P}^1 \longrightarrow \mathbb{P}^1/\mathcal{I}$ can be naturally identified with the cover $\pi_5 : (\Gamma(5)\backslash\mathbb{S}_1)^* \cong \mathbb{P}^1 \longrightarrow (\Gamma\backslash\mathbb{S}_1)^* \cong \mathbb{P}^1$, where $(-)^*$ denotes compactification.*

Furthermore, since $u = \infty$ corresponds to $J(\infty)$ (on $\Gamma\backslash\mathbb{S}_1$), we know that the 12 vertices of the icosahedron are ***the 12 cusps*** of $\Gamma(5)$.

The ***icosahedral equation*** is just $u = q(z_0 : z_1)$ (see (4.59)), or, after dehomogenizing,

$$\left((z^{20}+1) - 228(z^{15} - z^5) + 494 z^{10}\right)^3 + 1728 u z^5 (z^{10} + 11 z^5 - 1)^5 = 0, \tag{4.60}$$

and it follows from the fact that the Galois group of q is A_5 that

Lemma 4.2.13 *The icosahedral equation (4.60) is its own Galois resolvent.*

Geometrically we have by Proposition 4.2.12 the following diagram

$$\begin{tikzcd} \mathbb{S}_1 \arrow[rr, "J"] \arrow[dr, "J_5"'] & & \Gamma\backslash\mathbb{S}_1 \\ & \Gamma(5). \arrow[ur, "q"'] & \end{tikzcd} \tag{4.61}$$

Here J_5 is the “Hauptmodul fünfter Stufe”, and can be written in terms of τ as

$$J_5(\tau) = q^{2/5} \cdot \frac{\sum_{-\infty}^{\infty} (-1)^n q^{5n^2 - 3n}}{\sum_{-\infty}^{\infty} (-1)^n q^{5n^2 - n}} = q^{2/5} \frac{\theta_1(2\tau, q^5)}{\theta_1(\tau, q^5)}, \tag{4.62}$$

where $q = e^{2\pi i \tau}$ and θ_1 is the theta function with characteristic $(1, 0)$. The diagram (4.61) also gives a ***solution*** of the icosahedral equation: given $u \in \mathbb{P}^1$, set

$$z = J_5(\tau), \tag{4.63}$$

where τ is a solution of the modular equation (4.26). Then from $q \circ J_5 = J$, we have $q(z) = q(J_5(\tau)) = J(\tau) = u$, i.e., $q(z) = u$. A solution of (4.62) can also be given directly in terms of hypergeometric functions; we refer to [Kl3] and the references therein (in the new addition, references provided by Slodowy) for details and formulas.

4.2.7 Klein's solution

The solution suggested by Klein had an algebraic and a transcendental part. We sketch this briefly, as these are typical steps for any resolution.

Algebraic Part

Classical: Reduction of equation with solvable Galois group to a "pure equation" $x^n = X$	*Icosahedral*: Reduction of equation of fifth degree to solution of the "icosehedral equation" $\frac{H^3(x)}{1728 f^5(x)} = X$

Actually, Klein's method for dealing with this reduction is almost completely geometric, which was his objective in writing [Kl3]. First of all, if we are given a general quintic $P(x)$ as in (4.49), then by means of a Tschirnhaus transformation it can be put in the form (4.50), determining a point on the quadric Q (4.51). Let P_P denote this point; P_P determines two fibres of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, one in each ruling. The action of Σ_5 on Q induces an action of A_5 on each factor, the two being related by an outer automorphism, and the orbit of P_P consists of 120 points (in $\mathbb{P}^3$). Let L_P, M_P denote the two lines in $\mathbb{P}^3$ which are the fibres of Q which meet at P_P, on each of which, we have an action of A_5 on $\mathbb{P}^1$. We may consider the quotient of $\mathbb{P}^1$ by A_5 on each copy; each quotient morphism is a copy of q as in (4.58), which we denote by q_1 and q_2. Let z_1, z_2 be affine coordinates on the first and second copy of $\mathbb{P}^1$; for $p \in \mathbb{P}^4$, we have two icosahedral equations (cf. ***loc. cit.*** p. 184):

$$u_1(p) = \frac{H^3(z_1)}{1728 f^5(z_1)}, \quad u_2(p) = \frac{H^3(z_2)}{1728 f^5(z_2)}, \tag{4.64}$$

where the "icosahedral solvents" u are determined rational algebraically in terms of the coefficients of the given Hauptgleichung (4.50), α, β, γ and $\nabla = \sqrt{\Delta}$. This is done as follows.

Let $\mathcal{O}_\nu$, $\nu = 1, \ldots, 5$ be the five inscribed octahedra; let $f_\nu = \prod(z - \{\text{vertices of } \mathcal{O}_\nu\})$, $W_\nu = \mathbf{Hess}(f_\nu) = \prod(z - \{\text{vertices of dual cube}\})$ be the invariant forms (for the ν^{th} octahedral subgroup of $\mathcal{I}$) of degrees six and eight, respectively. Then, supposing $(y_1, \ldots, y_5)$ are the five roots of P, upon permutation of the roots, permuting the octahedra should result, i.e., (after some calculations), letting $z = z_0/z_1$ be an affine coordinate on one of the $\mathbb{P}^1$'s of Q,

$$y_\nu = mT(z)W_\nu(z) + 12 n f^2(z) f_\nu(z) W_\nu(z), \tag{4.65}$$

for affine parameters m, n and z being acted on by the icosahedral group. Also, letting $y_1, \ldots, y_5$ be the five roots of a quintic equation, lying on the quadric Q, it is possible to express the parameters z_1, z_2 of (4.64) in terms of $y_1, \ldots, y_5$: set (these are the expressions of Lagrange)

$$\begin{aligned}
p_1 &= y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \varepsilon^4 y_4, \\
p_2 &= y_0 + \varepsilon^2 y_1 + \varepsilon^4 y_2 + \varepsilon y_3 + \varepsilon^3 y_4, \\
p_3 &= y_0 + \varepsilon^3 y_1 + \varepsilon y_2 + \varepsilon^4 y_3 + \varepsilon^2 y_4, \\
p_5 &= y_0 + \varepsilon^4 y_1 + \varepsilon^3 y_2 + \varepsilon^2 y_3 + \varepsilon y_4.
\end{aligned}$$

Then, as Klein shows, we have the relations (here the line geometry on the Grassmann $\mathbf{G}(2,4) \subset \mathbb{P}^5$, the Klein quadric, is applied)

$$z_1 = -\frac{p_1}{p_2} = \frac{p_3}{p_4}, \quad z_2 = -\frac{p_2}{p_4} = \frac{p_1}{p_3}. \tag{4.66}$$

Given a Hauptgleichung (4.50), the equations (4.64) and (4.65) may be combined, and utilizing furthermore the expression of z in terms of line coordinates (4.66) for $q(z) = u$, the icosahedral equation associated to the given set of roots is derived. Since now the elementary symmetric functions of the y_ν are the coefficients α, β and γ of the equation (and determine ∇), one gets a system of equations which can be solved rationally for m, n and u (as functions of $\alpha, \beta, \gamma, \nabla$). The results are:

$$\begin{aligned} m &= \frac{(11\alpha^3\beta + 2\beta^2\gamma - \alpha\gamma^2) \pm \alpha\nabla}{24(\alpha^4 - \beta^3 + \alpha\beta\gamma)} \\ u &= \frac{(48\alpha m^2 - 12\beta m - \gamma)^3}{64\alpha^2(12(\alpha\gamma - \beta^2)m - \beta\gamma)} \\ n &= -\frac{96\alpha m^3 + 72\beta m^2 + 6\gamma m - 12\alpha^2 u}{144\alpha m^2 + 12\beta m + \gamma}. \end{aligned} \tag{4.67}$$

The second equation gives the parameter u for an icosahedral equation $q(z_0 : z_1) = u$ (the two possible values correspond to the two icosahedral equations (4.64)); inserting a solution $z = z_0/z_1$ of (4.64) into (4.65), as well as the values of m, n from (4.67), gives an explicit formula for the roots of the given Hauptgleichung.

Transcendental Part

Classical: Solution of the "pure equation" by means of logarithims: $x = e^{\frac{1}{n} log X}$	*Icosahedral*: Solution of the icosahedral equation by means of elliptic functions: $x = q^{\frac{2}{5}} \frac{\vartheta_1(\frac{2\pi i K'}{K}, q^5)}{\vartheta_1(\frac{\pi i K'}{K}, q^5)}$

After determining the parameter $u = u(\alpha, \beta, \gamma, \nabla)$ for the given quintic equation, the corresponding icosahedral equation $q(z) = u$ can be solved by (4.63); the solution z (there are actually two solutions, corresponding to the two icosahedral equations (4.64)), together with values for m, n, u from (4.67), are inserted in (4.65). This gives the roots as sketched above. We summarize Klein's solution in the flow chart in Figure 4.10

4.3 Solving the equation of 27th degree for the 27 lines

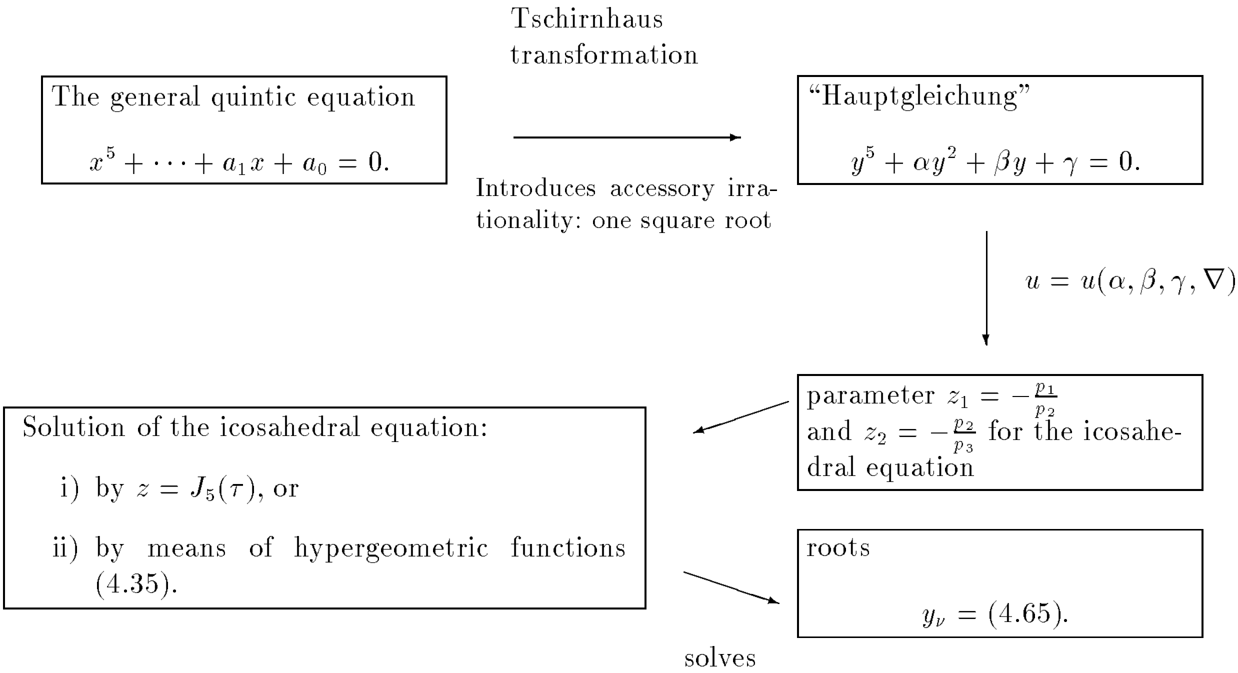
In [Kl1], Klein sketches in a letter to Jordan how one could use a similar process to solve the equation of 27^{th} degree determining the 27 lines on a smooth cubic surface. This was the starting point of investigations of Maschke [Ma2] and Burkhardt [Bu] in Göttingen. They were students of Klein, who, following suggestions of [Kl1] and Jordan, worked on the possibility of solving the equation of 27^{th} degree for the lines of the cubic surface by using hyperelliptic functions (nowadays theta functions of two variables). Hence these theta functions replace the theta functions of one variable used in the case of a quintic equation.

Start with the following nine theta functions with characteristics (see Definition 3.3.2):

$$X_{\alpha\beta} = \Theta \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{3} & \frac{\beta}{3} \end{bmatrix} (\tau, z), \ \tau \in \mathbb{S}_2, \ z \in \mathbb{C}^2, \ \alpha, \ \beta \in \mathbb{Z}/3\mathbb{Z}. \tag{4.68}$$

The $X_{\alpha\beta}$ have the property that transformation of τ (z being held fixed) by an element $\gamma \in P\Gamma(3)$ (the principal congruence subgroup of level three in $PSp(4, \mathbb{Z})$) induces a ***linear*** transformation of the

Figure 4.10: Klein's solution



$X_{\alpha\beta}$, yielding a linear action of $G_{25,920}$ on $\mathbb{P}^8$. This action splits into two invariant subspaces, namely:

$$\begin{array}{l} \mathbb{P}^3 = \{Z_{\alpha\beta} = \frac{1}{2}(X_{\alpha\beta} - X_{-\alpha-\beta})\}, \\ \mathbb{P}^4 = \{Y_{\alpha\beta} = \frac{1}{2}(X_{\alpha\beta} + X_{-\alpha-\beta})\}. \end{array} \tag{4.69}$$

We get a covariant action of the group $G_{25,920}$: let $\alpha \in G_{25,920}$ be given. Then as $\alpha \in PSp(4, \mathbb{Z}/3\mathbb{Z})$, a representative $\widetilde{\alpha}$ of α acts on $\tau \in \mathbb{S}_2$ (which is independent of the representative modulo $\Gamma(3)$), while at the same time α acts on the theta functions, hence on the $\mathbb{P}^3$ and the $\mathbb{P}^4$. Klein's basic idea was to consider the invariant $\mathbb{P}^3$, the "problem of the Z". Then the solution could be modeled on that of the quintic equation in the following sense. The icosahedral equation, concerning the cover $\Gamma(5)\backslash\mathbb{S}_1 \longrightarrow \Gamma\backslash\mathbb{S}_1$, is replaced in this situation by the map

$$\Gamma(3)\backslash\mathbb{S}_2 \longrightarrow \Gamma(1)\backslash\mathbb{S}_2.$$

However, at the time the space $\Gamma(1)\backslash\mathbb{S}_2$ was not known, so its use was bypassed by using invariants of what the spaces parameterize, here genus 2 curves. Since such invariants are forms they give maps to projective spaces, and a corresponding form problem can be formulated. The geometry here is more challanging than in the case of the icosahedron, for example the Hesse pencil of elliptic curves appears. In order to proceed along the same lines as was done in the case of the quintic, one first needs an understanding of the representations of $G_{25,920}$ in $\mathbb{P}^3$ and $\mathbb{P}^4$; in particular, the invariants should be calculated. We proceed to discuss these matters briefly, most of this material can be extracting from [Bu] or [Ma2].

4.3.1 The unitary reflection groups of order 25,920

We have actions of $G_{25,920}$ on $\mathbb{P}^3$ and on $\mathbb{P}^4$, both of which in fact are generated by unitary reflections. Such actions have ***invariant forms*** (or subvarieties) $f_i(x)$, where the variables x are homogenous

Table 4.2: The arrangement in $\mathbb{P}^3$ defined by $G_{25,920}$

space	k	$t_4(1)$	$t_2(1)$	t_5	t_{12}
#	40	90	240	360	40
$t(2)$	1	9	12	45	12
$t_4(1)$		1	0	4	4
$t_2(1)$			1	6	2

Table 4.3: The arrangement in $\mathbb{P}^4$ defined by $G_{25,920}$

space	k	$t_2(2)$	$t_3(2)$	$t_3(1)$	$t_4(1)$	$t_6(1)$	$t_9(1)$	t_4	t_{10}	t_{12}	t_{18}
#	45	270	240	270	720	540	40	540	216	45	40
$t(3)$	1	12	16	18	64	72	8	84	48	12	16
$t_2(2)$		1	0	3	8	6	0	18	12	3	4
$t_3(2)$			1	0	3	9	2	9	9	3	7
$t_3(1)$				1	0	0	0	6	0	2	0
$t_4(1)$					1	0	0	3	3	0	1
$t_6(1)$						1	0	1	2	1	2
$t_9(1)$							1	0	0	0	4

variables of the corresponding projective space, the determination of which is the area of invariant theory. The ***form problem*** now is as follows. Given ratios for the ***values*** of the invariant forms f_i, determine explicitly ratios for the values x_i of the variables (i.e., a point in the projective space). See the appendix, section B.1.2 for more details from a general point of view. Klein and his students studied the two particular representations in the 1890's. Maschke considered the action on $\mathbb{P}^3$; Burkhardt worked on $\mathbb{P}^4$. Tables for these arrangements, taken from [OS1], are given in Table 4.2 and Table 4.3. The first rows of these tables give the numbers of each type of subspace; the other rows give the inclusions, where we are using the notations of (3.4) for the subspaces.

The arrangement in $\mathbb{P}^3$: The arrangement induced in each of the 40 planes of this arrangement is the ***extended Hesse pencil***, the arrangement of 21 lines which are the 12 lines of the Hesse pencil together with the nine lines joining corners of the four triangles. The 12 lines of the Hesse pencil are the 12 two-fold lines lying in the plane, the nine other lines are the four-fold lines lying in the plane. We might also remark here that the five-fold points of the arrangement split into two different types of singular points in the planes, namely 36 two-fold points and nine four-fold points which are the base points of the Hesse pencil. The 12-fold points of the arrangement lying in one of the planes are five-fold points of the induced arrangement. For each 12-fold point we can also speak of the induced arrangement, by blowing up the point in $\mathbb{P}^3$ and considering the proper transforms of the 12 planes intersecting the point in the exceptional $\mathbb{P}^2$. In our case we get the Hesse pencil itself, i.e., the 12 lines of the four degenerate cubics.

The arrangement in $\mathbb{P}^4$: This arrangement is exactly the dual of the arrangement of the 45 Jordan primes which we will be discussing in the next chapter, of relavence to the Burkhardt quartic.

For example, the 40 18-fold points correspond to the 40 Steiner primes containing 18 of the 45 nodes, the 45 12-fold points correspond to the 45 Jordan primes, each containing 12 of the nodes, etc. We will be discussing this arrangement in great detail in the next chapter.

We now describe briefly the invariant forms under these representations. The invariant forms under the action of $G_{25,920}$ on $\mathbb{P}^3$ were calculated by Maschke in [Ma2]. This is done essentially by reducing the problem to that of the invariants of the Hesse group of order 648 acting on $\mathbb{P}^2$, as follows. The 40 planes of the arrangement in $\mathbb{P}^3$ are given explicitly in homogenous coordinates $(z_0 : \ldots : z_3)$:

$$\begin{array}{ll} (4) & z_i = 0 \qquad (i = 0, \ldots, 3) \\ (9) & (z_1^3 + z_2^3 + z_3^3)^3 - 27z_1^3z_2^3z_3^3 = 0 \\ (9) & (z_0^3 + z_1^3 + z_3^3)^3 - 27z_0^3z_1^3z_3^3 = 0 \\ (9) & (z_0^3 + z_1^3 + z_2^3)^3 - 27z_0^3z_1^3z_2^3 = 0 \\ (9) & (z_0^3 + z_2^3 + z_3^3)^3 - 27z_0^3z_2^3z_3^3 = 0 \end{array} \tag{4.70}$$

In the $\mathbb{P}^2$ given by $z_0 = 0$ with homogenous coordinates (z_1, z_2, z_3) the action of G_{648} is generated by five collineations $A, B, C, D, E : z \mapsto z'$,

$$\begin{array}{c|c|c|c|c|c} & A & B & C & D & E \\ \hline z_1' = & z_2 & z_1 & z_1 & z_1 & z_1 + z_2 + z_3 \\ z_2' = & z_3 & z_3 & \varrho z_2 & \varrho z_2 & z_1 + \varrho z_2 + \varrho^2 z_3 \\ z_3' = & z_1 & z_2 & \varrho^2 z_3 & \varrho z_3 & z_1 + \varrho^2 z_2 + \varrho z_3 \end{array} \tag{4.71}$$

A complete system of invariants for G_{648} has degrees 6, 9, 12, 12 and 18 and can be given explicitly by the following forms:

$$\begin{array}{rcl} C_6 & = & z_1^6 + z_2^6 + z_3^6 - 10(z_1^3z_2^3 + z_2^3z_3^3 + z_3^3z_1^3), \\ C_9 & = & (z_1^3 - z_2^3)(z_2^3 - z_3^3)(z_3^3 - z_1^3), \\ C_{12} & = & (z_1^3 + z_2^3 + z_3^3)[(z_1^3 + z_2^3 + z_3^3)^3 + 216z_1^3z_2^3z_3^3], \\ \mathfrak{C}_{12} & = & z_1z_2z_3[27z_1^3z_2^3z_3^3 - (z_1^3 + z_2^3 + z_3^3)^3], \\ C_{18} & = & ((z_1^3 + z_2^3 + z_3^3)^6 - 540z_1^3z_2^3z_3^3((z_1^3 + z_2^3 + z_3^3)^3) - 5832z_1^6z_2^6z_3^6. \end{array} \tag{4.72}$$

C_{18} is the functional determinant of C_{12} and $\mathfrak{C}_{12}$, $\mathfrak{C}_{12}$ is just the product of the 12 lines of the Hesse arrangement. C_{12} is the Hessian of $\mathfrak{C}_{12}$ and vise versa. C_9 is the so-called difference product of z_1^3, z_2^3, z_3^3.

The two relations among these invariants are:

$$\begin{array}{rcl} 432C_9^2 & = & C_6^3 - 3C_6C_{12} + 2C_{18}, \\ 1728\mathfrak{C}_{12}^3 & = & C_{18}^2 - C_{12}^3. \end{array} \tag{4.73}$$

Maschke proves that the action of $G_{25,920}$ in $\mathbb{P}^3$ is generated by the action of a G_{648} acting in one of the 40 planes and a tetrahedral group of order 24 consisting of the permutation group acting on the z_i. This tetrahedral group stabilizes the tetrahedron consisting of the first four forms in (4.70), and $G_{25,920}$ is generated by it and the G_{648} acting on $z_0 = 0$ as above. From this one deduces that each invariant form of $G_{25,920}$ can be written as a polynomial in z_0 with coefficients which are polynomial expressions in the invariants of G_{648}. The result is as follows. There are invariant forms of degrees 12, 18, 24, 30 and 40. F_{40} is just the product of the 40 planes defining the arrangement in $\mathbb{P}^3$. For example F_{12} is:

$$\begin{array}{rcl} F_{12} & = & 6z_0^{12} + 6\cdot 22C_6z_0^6 + 6\cdot 220C_9z_0^3 + C_6^2 + 5C_{12} \\ & = & 6\{\Sigma z_i^{12} + 22\Sigma z_i^6z_j^6 + 220\Sigma \pm z_i^6z_j^3z_k^3\}, \end{array} \tag{4.74}$$

where there are rules for determining which sign $\pm$ is applied. The forms F_{12}, F_{18}, F_{24}, F_{30} and F_{40} satisfy the (surprisingly complicated) relation,

$$2^{28}\cdot 3^{15}\cdot 5^{15}\cdot F_{40}^3=\begin{vmatrix}\Phi_{30} & 2\Phi_{24}^2 & F_{18}\Phi_{24} & F_{12}\Phi_{24}\\ 2\Phi_{24} & 27F_{12}\Phi_{30}-11F_{18}\Phi_{24} & 3F_{12}\Phi_{24}-4F_{18}^2 & 3\Phi_{30}-4F_{12}F_{18}\\ F_{18} & 3F_{12}\Phi_{24}-4F_{18}^2 & 13F_{12}F_{18}-3\Phi_{30} & 9F_{12}^2-2\Phi_{24}\\ F_{12} & 3\Phi_{30}-4F_{12}F_{18} & 9F_{12}^2-2\Phi_{24} & F_{18}\end{vmatrix} \tag{4.75}$$

where $4\Phi_{24}=25F_{24}-9F_{12}^2$, $6\Phi_{30}=25F_{30}-F_{12}F_{18}$. There are two simpler relations involving F_{40}. First, F_{40} is the Hessian determinant of F_{12}, and secondly the Jacobian determinant $\mathfrak{J}(F_{12},F_{18},F_{24},F_{30})=|\frac{\partial F_j}{\partial z_i}|$ satisfies (see Theorem B.1.14, (iii)):

$$\mathfrak{J}(F_{12},F_{18},F_{24},F_{30})=2^{26}\cdot 3^{15}\cdot 5^{6}\cdot F_{40}^2.$$

Burkhardt proceeded similarly to determine the invariants of $G_{25,920}$ acting on $\mathbb{P}^4$. Let the coordinates be given as

$$y_0=Y_{00},\ y_1=Y_{10},\ y_2=Y_{01},\ y_3=Y_{11},\ y_4=Y_{12} \tag{4.76}$$

where the $Y_{\alpha\beta}$ are the theta functions of (4.69). Using the isomorphism $G_{25,920}\cong PSp(4,\mathbb{Z}/3\mathbb{Z})$, it suffices to use generators of the symplectic group $PSp(4,\mathbb{Z}/3\mathbb{Z})$ to get generators of the action of $G_{25,920}$ on $\mathbb{P}^4$. However, Burkhardt used instead the corresponding hyperelliptic curves, and a certain ***Weierstraß form*** for them to describe the level 3 structure. At any rate, generators of the group are transformations B, C, D and S_2, which act as in the following table:

	B	C	D	S_2
$y_0'=$	$\frac{1}{\sqrt{-3}}(y_0+2y_1)$	y_0	$-y_0$	y_0
$y_1'=$	$\frac{1}{\sqrt{-3}}(y_0-y_1)$	y_1	$-y_2$	$\varrho^2 y_1$
$y_2'=$	$\frac{1}{\sqrt{-3}}(y_2+y_3+y_4)$	y_4	$-y_1$	y_2
$y_3'=$	$\frac{1}{\sqrt{-3}}(y_2+\varrho y_3+\varrho^2 y_4)$	y_2	$-y_3$	$\varrho^2 y_3$
$y_4'=$	$\frac{1}{\sqrt{-3}}(y_2+\varrho^2 y_3+\varrho y_4)$	y_3	$-y_4$	$\varrho^2 y_4$

(4.77)

Consider the hyperplane $\mathcal{S}$ given by $\mathcal{S}=\{y_0=0\}$ and the plane $\mathfrak{J}=\{y_0=y_1=0\}$. The stabiliser of each is a subgroup of order 648, but these two subgroups of order 648 are ***not*** conjugate to each other; indeed,

$$N(\mathcal{S})=\langle C,D,S_2\rangle,\quad N(\mathfrak{J})=\langle B,C,S_2\rangle. \tag{4.78}$$

Hence, just as above, $G_{25,920}$ is generated by the subgroup of order 648 acting on $\mathfrak{J}$, generated by B, C and S_2, and by the centraliser of $\mathfrak{J}$. Burkhardt shows this to be a homogenous tetrahedral group, and its invariants are:

$$\begin{aligned}\Phi &= y_0^4+8y_0y_1^3;\\ \Psi &= y_0^3y_1-y_1^4;\\ t &= y_0^6-20y_0^3y_1^3-8y_1^6;\end{aligned} \tag{4.79}$$

which satisfy the relation

$$\Phi^3-64\Psi^3=t^2. \tag{4.80}$$

Furthermore, letting $\varphi=y_2y_3y_4$, $\psi=y_2^3+y_3^3+y_4^3$ and $u=y_0\psi+6y_1\varphi$, all invariants of $G_{25,920}$ can be written as linear combinations of the following: Φ, $u;,t$, Ψ_1, C_6; C_9; Φ_3; t_3, C_{12}; C_{18}, where the expressions C_m are the invariants (4.72), the Φ, u, and t were just defined, and

$$\begin{aligned} \Psi_1 &= \psi(-y_0^3+4y_1^3)+18\varphi y_0^2 y_1; \\ \Phi_3 &= -\psi^3 y_0 + 18\varphi\psi^2 + 108\varphi^3 y_0; \\ t_3 &= \psi^3(y_0^3+8y_1^3) - 54\varphi\psi^2 y_0^2 y_1 + 324\varphi^2\psi y_0 y_1^2. \end{aligned} \tag{4.81}$$

Burkhardt then calculates the following expressions. There are invariants of degrees 4, 6, 10, 12, 18 and 45. The invariant of degree 45 is just the product of the 45 reflection hyperplanes defining the arrangement in $\mathbb{P}^4$. As to the others, Burkhardt gives the following expressions:

$$J_4 = \Phi + 8u \tag{4.82}$$

$$J_6 = t + 20\Psi_1 - 8C_6 \tag{4.83}$$

$$J_{10} = \frac{1}{24}(\Phi\Psi_1 + ut + 2\Phi C_6 + 2u\Psi_1 - 2\Phi_3 - 2uC_6) \tag{4.84}$$

$$\begin{aligned} J_{12} = {} & \frac{1}{24}(3t\Psi_1 + 3u\Phi^2 + 19\Psi_1^2 - 9u^2\Phi - 10C_6 t - 11t_3 + 9u^3 \\ & -2C_6\Psi_1 - 4C_{12} + 4C_6^2) \end{aligned} \tag{4.85}$$

and a similar expression for J_{18}. There is a relation between these, which takes the form J_{45}^2= rational expression in the other invariants, which Burkhardt does not explicitly calculate. There is also an invariant of degree 40: the product of the 40 Steiner primes. Denoting this by J_{40}, the following relation holds:

$$3^{33}J_{40} = [J_4^2(2^9J_{12} - J_4^3) - 3.2^{18}J_{10}^2]^2 - 2^{19}[J_4J_6 - 3.2^8J_{10}][J_4^3J_{18} - 2^{11}J_{10}^3]. \tag{4.86}$$

Coble mentions in [C1], III, p. 350 that “the expressions in y for the invariants J_4, J_6, J_{10}, J_{12}, and J_{18} given by Burkhardt seem open to suspicion”, and he gives other formulas for the invariants. He uses the fact, already noticed by Burkhardt, that the 40 ***squares*** of the linear forms defining the Steiner primes are permuted among each other, so that taking the sum of the k^{th} powers of these 40 squares is either an invariant of degree $2k$ or zero. Coble then shows that for $k = 2, 3, 5, 6, 9$ the sum of the k^{th} powers does not vanish, and hence, this yields a system of invariants. In fact, letting ξ_i be the 40 forms defining the 40 Steiner primes, it is clear that $I_k := \sum_{i=1}^{40}\xi_i^{2k}$ is an invariant if it does not vanish. In order for I_k, $k = 2, 3, 5, 6, 9$ to be non-vanishing it is sufficient to show that the Jacobian determinant does not vanish. This is what Coble shows, a calculation he simplifies by taking a convenient choice of coordinates.

4.3.2 Klein's suggestion

We sketch the method suggested by Klein, which was in many respects completed by Burkhardt in [Bu], III, briefly here.

Algebraic Part: Reduction to the “Maschke form problem”:

$$\frac{F_{24}(z)}{F_{12}^2(z)} = \alpha, \quad \frac{F_{30}(z)}{F_{12}(z)F_{18}(z)} = \beta, \quad \frac{F_{18}^2(z)}{F_{12}^3(z)} = \gamma.$$

Step 1: First, by (B.51) of the appendix we have the equation of a covariant surface, whose intersection with the given cubic is the set of 27 lines. By elimination theory this is reduced to a degree 27 equation in one variable, with Galois group $W(E_6)$. By adjoining the square root of the discriminant of the cubic, the group is reduced to $G_{25,920}$. A slightly different way of viewing this is as follows. The 27 lines determine 27 points on the Grasmannian $\mathbf{G}(2,4)$, which is a quadric hypersurface in $\mathbb{P}^5$. Letting A, B be two generic $\mathbb{P}^4$'s in the $\mathbb{P}^5$, each intersects the quadric in a divisor, and letting λ, μ be parameters, the solution to the equation

$$\lambda A + \mu B = 0$$

for a point on the quadric may be viewed as a solution $(\lambda : \mu) \in \mathbb{P}^1$ of the equation in one variable mentioned above. The elimination theory corresponds to projecting the 27 points onto a line, by which the covariance is lost.

Step 2: The action of the Galois group on the 27 roots, say $x_1, \ldots, x_{27}$, as well as the action of $W(E_6)$ (or $G_{25,920}$) on $\mathbb{C}^6$ (and consequently on $\mathbb{P}^5$), are fixed. In order to ***covariantly*** associate to a given equation the roots to a point in $\mathbb{P}^5$, we need equivariance, i.e., for $\gamma \in W(E_6)$ (or $G_{25,920}$), $\zeta(x_i)$ the point in $\mathbb{C}^6$ associated with x_i,

$$\zeta(\gamma x_i) = \gamma\zeta(x_i).$$

Now consider the roots $x_1, \ldots, x_{27}$ as variables; the Galois group then acts on the space $\mathbb{C}^{27} = \langle x_1, \ldots, x_{27}\rangle$. On the other hand, $W(E_6)$ acts linearly on $\mathbb{C}^6$; recall the 27 linear functions $a_i,\ b_i,\ c_{ij},\ i < j, 1 = 1, \ldots, 6$ of (6.7) below. It is possible to find a linear map

$$\begin{array}{rcl} \phi : \mathbb{C}^{27} & \longrightarrow & \mathbb{C}^6 \\ \langle x_1, \ldots, x_{27}\rangle & \mapsto & \xi_i(x_1, \ldots, x_{27}) \end{array} \tag{4.87}$$

such that $\xi_i \equiv a_i$ $(i = 1, \ldots, 6)$, $\xi_{i+6} \equiv b_i$, $(i = 1, \ldots, 6)$ and $\xi_{k+12} \equiv c_{ij}$, $(k = 1, \ldots, 15)$ by setting ([Bu], §75):

$$\xi_i(x_1, \ldots, x_{27}) = 4x_i - 2C_i + N_i, \tag{4.88}$$

where C_i is the sum of roots corresponding to the ten lines meeting the given x_i while N_i is the sum of the other 16 roots (so in particular, $C_i + N_i = \sum_{j=1}^{27} x_j - x_i$ for all $i = 1, \ldots, 27$). One sees in fact without difficulty that ϕ has image in the $\mathbb{C}^6 \subset \mathbb{C}^{27}$ defined by the 45 relations $a_i + b_j + c_{ij} = 0$, $c_{ij} + c_{kl} + c_{mn} = 0$ given by the tritangents (see the relation (6.10) below). Furthermore, ϕ is clearly equivariant with respect to the action of $W(E_6)$.

Step 3: A linear change of coordinates now passes from the a_i, b_i, c_{ij} to the "line coordinates" a_{ij} ([Bu], p. 324):

$$\begin{array}{rcl} a_1 & \equiv & \varrho a_{12} - a_{13} - \varrho^2 a_{34} + a_{42}, \\ a_2 & \equiv & \varrho a_{12} - \varrho a_{13} - \varrho^2 a_{34} + \varrho^2 a_{42}, \\ a_3 & \equiv & \varrho a_{12} - \varrho^2 a_{13} - \varrho^2 a_{34} + \varrho a_{42}, \\ a_4 & \equiv & a_{14} - \varrho^2 a_{12} - a_{23} + \varrho a_{34}, \\ a_5 & \equiv & \varrho a_{14} - \varrho^2 a_{12} - \varrho^2 a_{23} + \varrho a_{34}, \\ a_6 & \equiv & \varrho^2 a_{14} - \varrho^2 a_{12} - \varrho a_{23} + \varrho a_{34}. \end{array}$$

and the corresponding inverse relations

$$\begin{array}{rcl} a_{12} & = & \frac{i}{3\sqrt{3}}(\varrho a_1 + \varrho a_2 + \varrho a_3 + \varrho^2 a_4 + \varrho^2 a_5 + \varrho^2 a_6), \\ a_{13} & = & -\frac{1}{3}(a_1 + \varrho^2 a_2 + \varrho a_3), \quad a_{14} = \frac{1}{3}(a_4 + \varrho^2 a_5 + \varrho a_6), \\ a_{34} & = & \frac{i}{3\sqrt{3}}(\varrho^2 a_1 + \varrho^2 a_2 + \varrho^2 a_3 + \varrho a_4 + \varrho a_5 + \varrho a_6), \\ a_{42} & = & \frac{1}{3}(a_1 + \varrho a_2 + \varrho^2 a_3), \quad a_{23} = -\frac{1}{3}(a_4 + \varrho a_5 + \varrho^2 a_6). \end{array}$$

By means of these one now gets ***rational functions of the roots*** $x_1, \ldots, x_{27}$ ***which are*** a_{ij} ***'s.*** This requires taking the a_{ij} as homogeneous coordinates, and further, assuming that

$$a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 0,$$

i.e., the point $\zeta(x_1, \ldots, x_{27}) \in \mathbb{P}^5$ defined by ϕ (for a fixed equation of degree 27) actualy lies on the Klein quadric, see [Bu], p. 326 for details.

Step 4: Reduction from the problem of the a_{ik} to that of the $z \in \mathbb{P}^3$. There is a general method to do this, developed by Klein for his studies of the equations of seventh and eighth degree. The idea is to use the fact that the line coordinates a_{ik} describe variable lines in $\mathbb{P}^3$, so, given two such lines which intersect, we get a point in $\mathbb{P}^3$. So suppose we could, from the given point in $\mathbb{P}^5$, get the equations for two covariant lines which ***intersect***; then the point of intersection is a covariantly associated point. To do this explicitly, one uses the following fact: the most general kind of expression in $\mathbb{P}^5$ which is covariant is of the following form (let ζ_i denote the i^{th} coordinate of $\zeta(x_1, \ldots, x_{27}) \in \mathbb{P}^5$, i=1,..., 6):

$$X_i = \lambda_1\zeta_i + \lambda_2\left(\zeta_i^2 - \frac{s_2}{6}\right) + \cdots + \lambda_6\left(\zeta_i^6 - \frac{s_6}{6}\right), \tag{4.89}$$

where X_i is the covariant expression in the roots x_i, s_k is given by the sum of the k^{th} powers of the ζ_i and we are assuming $s_1 = 0$. Now for this to describe a variable ***line***, this point must lie on the Grassmannian, in other words must fulfill a quadratic relation. Given two such, say X_i' and X_i'', they will determine lines which ***meet***, if, in addition the relation $\sum X_i'X_i'' = 0$ is fulfilled. So we set X_i' and X_i'' as in (4.89), where X_i' will have coefficients λ_i', X_i'' will have coefficients λ_i'', and require:

$$\sum X_i'^2 = 0, \quad \sum X_i'X_i'' = 0, \quad \sum X_i''^2 = 0; \tag{4.90}$$

the first and last equations are equations for variable lines (i.e., that the covariant points lie on the Grassmannian), while the second expresses the condition that these lines meet. Inserting now the expressions (4.89) for X_i' and X_i'' into (4.90), one gets ***quadratic equations*** for the coefficients λ_i' and λ_i'', which can be solved. These expressions are then accessory irrationalities, necessary for the solution. At any rate we now have the variable covariant points X_i' and X_i''. To get the corresponding z (given by the intersection of the lines which X_i' and X_i'' define), note that a general line will meet both X_i' and X_i'' (and hence also their intersection point) if

$$\sum X_i'\xi = \sum X_i''\xi = 0. \tag{4.91}$$

If we now express that general line ξ in terms of the a_{ik}, then the equation (4.91) becomes a system of linear equations in the a_{ik}, and if one utilizes finally the relation

$$a_{ij} = z_i z_j' - z_i' z_j$$

we get a system of equations linear in two sets of variables, z_i and z_i'. The ***solution*** is then the point z for which (4.91) holds ***for all*** z_i'. For more details see the Abh. LVIII, §§8-11 in Klein's collected papers.

The solution z from the last step gives the ***values*** for the Maschke form problem, as a rational expression of the coefficients and two accessory square roots introduced in Step 4, and the reduction to this problem has been done.

Transendental Part: Solution of the Maschke equation in terms of the hyperelliptic functions $Z_{\alpha,\beta}$ of (4.69)

$$Z_{\alpha\beta} = \frac{1}{2}(X_{\alpha\beta} - X_{-\alpha-\beta}).$$

Once again there are several steps involved.

Step 5: The first observation is that the invariants of the Maschke group, $F_{12}, \ldots, F_{40}$, are ***invariants for a corresponding binary sextic***, the roots of which define a genus 2 curve, hence an abelian surface as Jacobian, whose theta functions will give the solution. Furthermore, by expressing the $Z_{\alpha\beta}$ in terms of $\tau \in \mathbb{S}_2$ (the Siegel space of degree two), one also gets expressions for the $F_{12}, \ldots, F_{40}$ in terms of τ. If one uses the Fourier expansions as usual, then setting $p := \exp(2\pi i\tau_{11})$, $q := \exp(2\pi i\tau_{12})$, $r := \exp(2\pi i\tau_{22})$, where we are writing $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}$, and setting $s := q - q^{-1}$, Burkhardt finds ([Bu] III, p. 328)

$$\begin{array}{rcl} p_{12}^{6} \cdot F_{12} & = & 6p\{1 - 12p^2 + 176r^2 + \cdots\} \\ p_{12}^{9} \cdot F_{18} & = & 54p^{3/2}\{-1 + 18p^2 + 136r^2 + \cdots\}, \\ p_{12}^{12} \cdot F_{24} & = & 1728p^2\{8r^2 + \cdots\}, \\ p_{12}^{15} \cdot F_{30} & = & 2592p^{5/2}\{-8r^2 + \cdots\} \\ p_{12}^{20} \cdot F_{40} & = & 8p^4r^4\{-2 - 4rs - 38p^2 - 2r^2 - s^2 + \cdots\}, \end{array} \tag{4.92}$$

where the expressions $\cdots$ are of third or higher degree. Here

$$p_{12} = \tau_{11}\tau_{22} - \tau_{12}^2$$

is the determinant of the period.

Step 6: Given a binary sextic $f(x_1, x_2) = 0$, a level 2 structure on the genus 2 (hyperelliptic) curve described by it amounts to an ordering of the branchpoints (see 2.4.7). The sextic may be described in ten ways $f = \phi_3\psi_3$ as the product of two cubics, corresponding to the ten even thetas with $\frac{1}{2}$-characteristics, and in six ways $f = \phi_1\psi_5$ as the product of a linear and quintic term, corresponding to the six odd characteristics. As the $Z_{\alpha\beta}$ are ***even*** functions for ***odd*** characteristics (see [Bu] II, p. 184), for the problem of the Z's one considers the latter type of decomposition:

$$f = \phi(x_1, x_2)\psi_5(x_1, x_2). \tag{4.93}$$

Then invariants of f are ***simultaneous invariants*** (see Definition B.1.4) of ϕ and ψ_5. In order to express the $F_{12}, \ldots, F_{40}$ in terms of the coefficients of f in (4.93), Burkhardt determines the corresponding degrees, using the fact that the complete system of invariants for sextics is known (see (B.29)). Using the power series expansion (4.92) and the discriminant D of the quintic ψ_5 in (4.93), he finds

$$\begin{array}{rcl} F_{12} & = & D^{1/2} \cdot J_{6,2}, \\ F_{18} & = & D^{3/4} \cdot J_{9,3}, \\ F_{24} & = & D \cdot J_{12,4}, \\ F_{30} & = & D^{5/4} \cdot J_{15,5}, \\ F_{40} & = & D^2 \cdot J_{20,8}. \end{array}$$

Here $J_{\kappa,\lambda}$ is a common invariant of ϕ and ψ_5, of degree κ in the coefficients of ϕ and degree λ in the coefficients of ψ_5. Without much difficulty he determines the $J_{\kappa,\lambda}$:

$$\begin{array}{rcll} J_{6,2} & = & \text{Hessian of } \psi_5 = [H], & \\ J_{9,3} & = & \mathfrak{J}(J_{6,2}, \psi_5) = [T], & \\ J_{12,4} & = & \psi_5^2 \cdot (\psi_5\psi_5)^4 = [\psi_5^2 i] & \begin{pmatrix} (\psi_5\psi_5)^4 =: i \text{ is the fourth transvec-} \\ \text{tion of } \psi_5 \text{ over itself, see (B.15)} \end{pmatrix} \\ J_{15,5} & = & \psi_5^2 \cdot ((\psi_5\psi_5)^4\psi_5)^1 = [\psi_5^2(i\psi_5)^1] & \quad (\ )^1 \text{ is again the transvection} \\ J_{20,8} & = & [\psi_5^4]. & \end{array} \tag{4.94}$$

In each case the bracket [ ] indicates that the variable in the expressions on the right hand side are replaced by the coordinates of the zero of ϕ. In particular, for the so-called Weierstraß form

$$\phi = x_2, \quad \psi_5 = 4x_1^4 - g_2 x_1^3 x_2^2 - g_3 x_1^2 x_2^2 - g_4 x_1 x_2^4 - g_5 x_2^5, \tag{4.95}$$

one has

$$\begin{aligned} [H] &= -\frac{2}{5} g_2, \\ [T] &= -\frac{8}{5} g_3, \\ [i] &= -\frac{4}{5} g_4 + \frac{3}{100} g_2^2, \\ [(i\psi_5)] &= -16 g_5 + \frac{2}{25} g_2 g_3, \\ [\psi_5] &= 4. \end{aligned} \tag{4.96}$$

Hence, once we are given the ***values*** of the Maschke form problem, the coefficients of an explicit sextic are given by (4.96).

Step 7: From the equation $f(x_1, x_2) = 0$, the non-normalized period Ω is given is given by:

$$\Omega = \begin{pmatrix} \int_{\alpha_1} \frac{x\,dx}{\sqrt{f}} & \int_{\alpha_2} \frac{x\,dx}{\sqrt{f}} & \int_{\beta_1} \frac{x\,dx}{\sqrt{f}} & \int_{\beta_2} \frac{x\,dx}{\sqrt{f}} \\ \int_{\alpha_1} \frac{dx}{\sqrt{f}} & \int_{\alpha_2} \frac{dx}{\sqrt{f}} & \int_{\beta_1} \frac{dx}{\sqrt{f}} & \int_{\beta_2} \frac{dx}{\sqrt{f}} \end{pmatrix}, \tag{4.97}$$

where as usual $\alpha_1, \alpha_2, \beta_1, \beta_2$ is a base of $H_1(C, \mathbb{Z})$, C the genus 2 curve defined by f. The normalized period τ defined by Ω is a solution of the Maschke form problem, i.e.,

$$\frac{F_{24}(z)}{F_{12}^2(z)} = \alpha, \quad \frac{F_{30}(z)}{F_{12}(z)F_{18}(z)} = \beta, \quad \frac{F_{18}^2(z)}{F_{12}^3(z)} = \gamma.$$

take on the given values.

This approach of Klein uses the representation of $G_{25,920}$ on $\mathbb{P}^3$, and not so much that on $\mathbb{P}^4$. But in fact one ***can*** use the representation in $\mathbb{P}^4$ instead of the above to solve the equation, a fact which was recognized somewhat later by Coble [C1]. Although this solution is by far more complicated, it is more canonical, and, moreover, it uses the action of $G_{25,920}$ on $\mathbb{P}^4$, which we also wish to study, so we will also sketch Coble's method.

4.3.3 Coble's solution

Coble's solution also consists of an algebraic and a trancendental part. We will sketch the algebraic part in this section, and the transcendental part will be discussed in the next chapter.
Algebraic part: Reduction of the problem of finding the equations for the 27 lines to the Burkhardt form problem (see Definition B.1.15 in the appendix) for the group $G_{25,920}$ acting on $\mathbb{P}^4$, which we will also refer to as "the Burkhardt group".

This reduction consists of several steps, one of which is quite complicated, one of which is quite easy.

Step 1: This is the easy part: reduction to the equation problem for $G_{6,2}$ acting on $\mathbf{P}_6^2$, see Definition 4.1.17.

At this point we note the $G_{6,2}$ acts only birationally on $\mathbf{P}^2_6$, so what we do here is only valid on a Zariski open set of the moduli space of cubic surfaces; but on the complement of this Zariski open set, the Galois group of the problem should shrink to a solvable subgroup (as was the case with the Cayley cubic). For the following, let $S \subset \mathbb{P}^3$ be given, say by $P = 0$, and we assume that $i_{100}(S) = I_{75}(S) \neq 0$ (if this does not hold, then the Galois group reduces to $W(F_4)$, which is solvable). *If* we know both the quatenary cubic $P(u)$, $u = (u_0, u_1, u_2, u_3)$ homogenous coordinates on $\mathbb{P}^3$, and the hexahedral form $\overline{a}, \ldots, \overline{f}$ of S, then the relation (B.55) implies that the coordinates u in $\mathbb{P}^3$ and $(a, \ldots, f)$ in hexahedral form are related by:

$$\begin{aligned} (l_{11}u) &= (L_8 a), \quad (l_{19}u) = (L_{14}a), \\ (l_{27}u) &= (L_{20}a), \quad (l_{43}u) = (L_{32}a), \end{aligned} \tag{4.98}$$

where the notation $(L_j a)$ means L_j is a linear function of $(a, \ldots, f)$. Also, for the sum $a + \cdots + f$ we will use the abbreviation $\sum a$, and similarly for other expressions as well as for the coefficients $\overline{a}, \ldots, \overline{f}$. For example, $\sum \overline{a}^3$ will mean $\overline{a}^3 + \cdots + \overline{f}^3$.

Given the cubic surface S, the quatenary invariants and the hexahedral invariants are related by equalities like $i_8(S) = I_6(S)$, etc. Consequently the following steps can be carried out and complete the reduction of Step 1.

1° For the given S the invariants $i_8(S), \ldots, i_{40}(S)$ and the linear covariants $(l_{11}u), \ldots, (l_{43}u)$ are calculated.

2° The values $i_8(S), \ldots, i_{40}(S)$ furnish the values for $I_6, \ldots, I_{30}$ of the equation problem for $G_{6,2}$. Assuming this problem has been solved, we have the coordinates (x, y, z, t, u) of a point in $\mathbf{P}^2_6$.

3° These coordinates can be inserted into (4.15), and the hexahedral form for the surface $(\overline{a}, \ldots, \overline{f})$ is derived from (4.16), while the value of d_2 is derived from (4.18).

4° The coefficients of the linear covariants $(L_8a), \ldots, (L_{32}a)$ are determined by the $\overline{a}, \ldots, \overline{f}$, so from the values calculated in 3°, the linear covariants can be calculated. Also, from the equations (4.13) and (4.14) the equations for the tritangents and lines are calculated, in terms of the $\overline{a}$.

5° By 1° we have the linear covariants $(l_j u)$, and by 4° we have the linear covariants $(L_j a)$. Hence, we can solve (4.98) for the coordinates a in terms of the u_i, assuming that the determinant of the four linear forms does not vanish. Thus we have the coordinates $a, \ldots, f$ which display S as a section of the Segre cubic, and the equations for the lines and tritangents in the u_i coordinates are derived by setting the expressions obtained for the $a, \ldots, f$ as linear expressions in the u_i in the equations for the lines and tritangents (4.13) and (4.14).

If the determinant of the four covariants used in 5° vanishes, this means that $I_{75} = 0$, which we have excluded. This completes the reduction to the equation problem for $G_{6,2}$.

Step 2: Translation of the ***rational*** action of $G_{6,2}$ on the hexahedral variables $a, \ldots, f$ into a ***linear*** action, that of the Burkhardt group.

Here one wants to give a relationship between coordinates on the space of coefficients $\overline{a}, \ldots, \overline{f}$ on the one hand, and coordinates y_i for the $\mathbb{P}^4$ with action of Burkhardt's group on the other. As we now have so many different coordinates, we pause to fix their notations.

Notations 4.3.1

- P_6^2 is the invariant space of six points in $\mathbb{P}^2$, birational to $\mathbb{P}^4$ via the map

$$\begin{array}{ccc} \mathsf{P}_6^2 & \longrightarrow & \mathbb{P}^4 \\ p & \mapsto & \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 1 & x & z \\ 0 & 1 & 0 & 1 & y & t \\ 0 & 0 & 1 & 1 & u & u \end{array}\right), \ (x,y,z,t,u) \in \mathbb{P}^4, \end{array}$$

  see (4.15).

- $\mathbb{P}^4$ with coordinates $(a,\ldots,f)$, $\sum a = 0$, with a linear action of Σ_6 and a rational action of $W(E_6)$. The dual coordinates are $\overline{a},\ldots,\overline{f}$, and the usual relation $\sum \overline{a}a = 0$ holds (see (B.6)).

- $\mathbb{P}^4$ with coordinates $(x_a,\ldots,x_f)$, $\sum x_a = 0$, with a linear action of $G_{25,920}$ (Burkhardt's group). There are contragredient variables $(u_a,\ldots,u_f)$, $\sum u_a = 0$. Here the sum $\sum x_a u_a$ is an invariant (see (4.108)).

- the same $\mathbb{P}^4$, but with coordinates y_i as in (4.76). Here the dual coordinates will be denoted v_i. In these variables there is once again an action of Burkhardt's group.

Remark 4.3.2 The transformation from the (x_a, u_a) to the (y_i, v_i) are given in (5.2) and (5.3) below.

Note in particular that the notations $x_1,\ldots,x_6$ used in our description of the Segre cubic are ***denoted*** here $a,\ldots,f$. Now recall the Cremona transformations defined by the Jacobian ideals of $\mathcal{S}_3$ and $\mathcal{I}_4$, which are the quadrics on ten points (see Corollary 3.2.4) and the cubics on 15 lines (see Lemma 3.3.13):

$$\varphi : \mathbb{P}^4 \longrightarrow \mathbb{P}^4 \text{ given by the Jacobian ideal of } \mathcal{S}_3 \tag{4.99}$$

$$d : \mathbb{P}^4 \longrightarrow \mathbb{P}^4 \text{ given by the Jacobian ideal of } \mathcal{I}_4. \tag{4.100}$$

In our situation, from the rational action of $G_{6,2}$ on $\mathbb{P}^4$ with coordinates (x,y,z,t,u), we get a rational action of $G_{6,2}$ on $\mathbb{P}^4$ with coordinates $a,\ldots,f$, $\sum a = 0$. Coble then shows how one can, guided by the invariants associated with the 27 lines, turn this into a linear action (of $G_{25,920}$) on $\mathbb{P}^4$ with coordinates $x_a,\ldots,x_f$. This is given by

Theorem 4.3.3 ([C1], III §3) *Under the duality map d of (4.100), the rational action of $G_{25,920}$ is transformed into the linear action of the Burkhardt group on $\mathbb{P}^4$.*

Before we sketch the proof let us pause a moment to see why this is natural. Recall that P_6^2 could be identified with the double cover of $\mathbb{P}^4$ branched along the Igusa quartic $\mathcal{I}_4$ (Theorem 3.5.4), hence also with 15 singular lines. On the other hand, $\mathcal{I}_4$ itself represented the condition on six points that they lie on a conic, while the 15 lines represented the condition that three of the points are on a line, so that under a ***regular*** action of $W(E_6)$, these loci must be exchangeable with the Igusa quartic, i.e., must be divisors isomorphic to (the proper transform of) $\mathcal{I}_4$ on some modification of P_6^2. This means that to get a regular action we should blow up the 15 lines, and that is precisely what the map d does.

Remark 4.3.4 There is another rather remarkable interpretation of the maps d and φ, also due to Coble (see [C2]). For this we first describe two normal forms for binary sextics (compare also (4.95)).

1) The Maschke normal form:

$$\Phi = \prod(y - \Phi_i) = y^6 - 6F_8y^4 + 4F_{12}y^3 + 9F_8^2y^2 - 12F_{20}y + 4F_{24} = 0.$$

2) The Joubert normal form, given in (B.36):

$$S = \prod(X - Q_i) = X^6 + 15q_2X^4 + 15q_4X^2 + 6q_5X + q_6 = 0.$$

In the situation at hand, the sextic in Joubert normal form is of the following special type:

$$\begin{aligned} T &= t^6 - 3\cdot 5\cdot 2^7A_0t^4 + 3\cdot 5(5\cdot 4^7A_0^2 - 9B_0)t^2 \\ &\quad + \frac{\sqrt{\Delta_0}}{8}t - 10(5^2\cdot 4^{10}A_0^3 - 5\cdot 12^3A_0B_0 - 3^3C_0) = 0. \end{aligned}$$

This is related to the Maschke normal form by the relations

$$A_0 = \frac{\rho^2}{2\cdot 5\cdot 2^7}\frac{F_{12}^2 - F_{24}}{F_8F_{12} - F_{20}}, \quad B_0 = \frac{\rho^4}{3^4\cdot 5}F_8, \quad C_0 = \frac{-\rho^6}{3^6\cdot 5}F_{12},$$

$$\Delta_0 = \frac{2^8\cdot\rho^{10}}{3^4}(F_8F_{12} - F_{20}), \quad \rho = \frac{(2\pi i)^2}{p_{12}}.$$

Here p_{12} is again the determinant of the period of the genus 2 curve defined by the sextic, and $F_4, \ldots, F_{20}$ are invariants defined by Maschke (which we have not defined). In fact, there is a Tschirnhaus transformation between the two normal forms. For a sextic in Joubert form a quadratic Tschirnhaus transformation leads to a sextic in Maschke normal form, and conversly, a sextic in Maschke normal form can be transformed by a cubic Tschirnhaus transformation into the Joubert normal form,

$$Q_i = -\Phi_i^3 + 3F_8\Phi_i - 2F_{12}, \tag{4.101}$$

where Φ_i are the roots of Φ and Q_i are the roots of the sextic in Joubert form. Then the incredible geometric interpretation is the following

Theorem 4.3.5 ([C2], (33)) *The quadratic and cubic Tschirnhaus tranformations just mentioned are inverses, mapping the Joubert normal form to the Maschke normal form and vica versa. Geometrically these are the birational maps φ and d given by the Jacobian ideals of the Segre cubic $\mathcal{S}_3$ and the Igusa quartic $\mathcal{I}_4$.*

That is, thinking of Q_i (respectively Φ_i) as homogenous coordinates on $\mathbb{P}^5$ with $\sum Q_i = 0$ (respectively on $(\mathbb{P}^5)^\vee$ with $\sum\Phi_i = 0$), then the birational maps φ and d are given by the Tschirnhaus transformations, φ by a quadratic one and d by the cubic one (4.101).

Sketch of proof of 4.3.3 Let $\sigma_i(\overline{a})$ be the elementary symmetric functions of the coefficients $\overline{a}, \ldots, \overline{f}$, and consider the Tschirnhaus transformation (4.101). Here this is:

$$\begin{aligned} \overline{\alpha} &= \sigma_3(\overline{a}) - \sigma_2(\overline{a})\overline{a} - 2\overline{a}^3, \\ \vdots\ &\ \vdots\ \ \vdots \\ \overline{\zeta} &= \sigma_3(\overline{a}) - \sigma_2(\overline{a})\overline{f} - 2\overline{f}^3, \end{aligned} \tag{4.102}$$

Then the usual relation $\overline{\alpha} + \overline{\beta} + \overline{\gamma} + \overline{\delta} + \overline{\varepsilon} + \overline{\zeta} = 0$ holds. Coble defines 40 irrational invariants of the cubic surface corresponding to the 40 complexes of triples of azygetic double sixes as described in (4.2) and (6.2). "Irrational invariant" means it is defined by homogenous polynomials with coefficients in the splitting field of the Galois extension defining the equation for the 27 lines and not in the field over which the cubic surface is defined. In terms of the six points given by (x, y, z, t, u) as in 4.3.1, these are given by expressions like (the notation (ijk) for the 3×3 minors of (3.85) was explained there)

$$\gamma_{123,456} = d_2(123)(456) = d_2(xt - yz + u(y - t - x + z)), \tag{4.103}$$

and so is of degree 3 in the coordinates of each point p_i. This yields $\frac{1}{2}\binom{6}{3} = 10$ such invariants. There are 30 others given by expressions

$$\gamma_{ij,kl,mn} = (ikl)(jkl)(kmn)(lmn)(mij)(nij),$$

once again cubic in the variables. For these one has

$$\gamma_{12,34,56} + \gamma_{12,56,34} = \overline{\alpha} + \overline{\delta},$$

and permutations. The important change of variables, now ***linear***, is given by

$$\begin{array}{l} x_a = \overline{\alpha} + (\rho - \rho^2)d_2\overline{a}, \ldots, x_f = \overline{\zeta} + (\rho - \rho^2)d_2\overline{f}, \\ u_a = \overline{\alpha} + (\rho^2 - \rho)d_2\overline{a}, \ldots, u_f = \overline{\zeta} + (\rho^2 - \rho)d_2\overline{f}, \end{array} \tag{4.104}$$

These again fulfill $x_a + \ldots + x_f = 0 = u_a + \ldots + u_f$, and the 40 invariants above take the form

$$\begin{array}{rcl} 2(\rho - \rho^2)\gamma_{123,456} & = & x_a + x_b + x_c - u_a - u_b - u_c, \\ 2(\rho - \rho^2)\gamma_{12,34,56} & = & \rho x_a - \rho^2 x_d - \rho^2 u_a + \rho u_d, \\ 2(\rho - \rho^2)\gamma_{12,56,34} & = & -\rho^2 x_a + \rho x_d + \rho u_a - \rho^2 u_d, \end{array} \tag{4.105}$$

and permutations of this. Moreover, he defines 45 ***tritangent invariants***, which have the property of vanishing when the surface defined by the six points aquires an Eckard point. They are given in terms of the x_a and u_a as follows:

$$\begin{array}{rcl} 2t_{12,34,56} & = & x_a - x_d, \\ 2\tau_{12,34,56} & = & u_a - u_d, \\ 2t_{1256,4} & = & -\rho(x_b + x_c) + \rho^2(x_e + x_f), \\ 2\tau_{1256,4} & = & -\rho^2(u_b + u_c) + \rho(u_e + u_f). \end{array} \tag{4.106}$$

Since the variables $(x_a, \ldots, x_f)$ and $(u_a, \ldots, u_f)$ are invaraint under permutations, to check that these coordinates describe the action of Burkhardt's group it suffices to determine what the action of the Cremona transformation A_{123} (which exchanges the double sixes N and N_{123}) is on these variables. It is easily checked that A_{123} acts on these variables as:

$$\begin{array}{rcl} 6x_a' & = & (-3\rho^2 + \rho)u_a + (3\rho^2 + \rho)(u_b + u_c) + (u_d + u_e + u_f), \\ 6x_b' & = & (-3\rho^2 + \rho)u_b + (3\rho^2 + \rho)(u_a + u_c) + (u_d + u_e + u_f), \\ 6x_c' & = & (-3\rho^2 + \rho)u_c + (3\rho^2 + \rho)(u_a + u_b) + (u_d + u_e + u_f), \\ 6x_d' & = & (u_a + u_b + u_c) + (-3\rho + \rho)^2 u_d + (3\rho + \rho^2)(u_e + u_f), \\ 6x_e' & = & (u_a + u_b + u_c) + (-3\rho + \rho)^2 u_e + (3\rho + \rho^2)(u_d + u_f), \\ 6x_f' & = & (u_a + u_b + u_c) + (-3\rho + \rho)^2 u_f + (3\rho + \rho^2)(u_d + u_e), \end{array} \tag{4.107}$$

Similarly, the images under A_{123} of the u_a can be expressed in terms of the x_a, with coefficients which are the conjugates of the coefficients above. As one sees from this, under the action of $W(E_6)$ on the space $\mathbb{P}^9 = \{x_a, u_a, \sum x_a = 0 = \sum u_a\}$, the element A_{123} exchanges the x and the u coordinates, so the two skew $\mathbb{P}^4$'s given by

$$\mathbb{P}^4 = \{u_a = \cdots = u_f = 0\}, \quad (\mathbb{P}^4)^\wedge = \{x_a = \cdots = x_f = 0\}$$

are projective spaces which are exchanged under ***odd*** elements of $W(E_6)$, but fixed by $G_{25,920}$. Furthermore, a calculation shows that

$$\sum_{40} \gamma_{123,456}^2 = 2\sum x_a u_a = 4I_6, \tag{4.108}$$

where I_6 is the invariant (B.56) of the hexahedral form which corresponds to the invariant i_8 of the quatenary cubic. From this it follows that under $G_{25,920}$, the variables x_a and u_a are contragredient, and odd elements of $W(E_6)$ exchange $\mathbb{P}^4$ and $(\mathbb{P}^4)^\wedge$, but have the invariant (4.108). This state of affairs is often referred to by saying $W(E_6)$ is a correlation group acting on $\mathbb{P}^4$, with $G_{25,920}$ as the invariant collineation group. The collineation group is now easily identified with the Burkhardt group, and 4.3.3 is established. □

The Burkhardt group will be studied more thoroughly in the next chapter. We just remark that in the coordiantes x_a, the Burkhardt invariant J_4 (4.82) becomes

$$J_4 = \sum x_a x_b x_c x_d, \quad \sum x_a = 0. \tag{4.109}$$

It is now a straightforward matter to relate the hexahedral invariants to the invariants of the Burkhardt group. This is done by utilizing (4.103) and (4.104) to change variables from the x_a to the $\overline{a}, \ldots, \overline{f}$. The result is

Theorem 4.3.6 ([C1], III (42)) *Any invariant of the Burkhardt group of total degree $2k$ in x and u, or the sum or difference of two dual invariants of total degree $2k$, is an invariant of the hexahedral form of degree $6k$ in $\overline{a}, \ldots, \overline{f}$, which is rational in $I_6, \ldots, I_{30}$.*

Step 3: Reduction to the Burkhardt form problem.

Now one must show how a solution of the Burkhardt form problem (see Definition B.1.15) gives a solution of the equation problem for $G_{6,2}$ (Definition 4.1.17). Let $p = (p_a, \ldots, p_f)$ be a solution, in the coordinates x_a, of the Burkhardt form problem. Then the factor of proportionality λ in the x_a is determined by the value of J_6/J_4, at least up to sign. It remains to transfer this solution to one in $\mathbf{P}_6^2$, i.e., to find the corresponding point (x, y, z, t, u). For this one requires linear covariants, as in (4.98) above. Let $J_{0,4}$ be the dual form of $J_4 = J_{4,0}$, where here the two indices indicate degree in the x_a and in the u_a. This is the invaraint of the Burkhardt group acting on the u variables of degree four; explicit formula in the y_i variables (see 4.3.1) and their dual coordinates v_i are

$$J_{4,0} = y_0^4 + 8y_0(y_1^3 + \cdots + y_4) + 48y_1y_2y_3y_4,$$
$$J_{0,4} = v_0^4 + v_0(v_1^3 + \cdots v_4^3) + 3v_1v_2v_3v_4.$$

Let $\mathcal{P}_{3,1} := \Delta_v^3 J_{0,4}$ be the cubic polar (Definition B.1.6, see also Lemma B.2.2). Then acting with $\mathcal{P}_{3,1}$ on any invariant $J_{r,0}$ yields a linear covariant:

$$\mathcal{P}_{3,1} \vdash J_{r,0} = J_{r-3,1}. \tag{4.110}$$

Proposition 4.3.7 ([C1], (45)) *The invariants $J_4, \ldots, J_{18}$ of the Burkhardt group yield by (4.110) linear covariants, linearly independent in v, and with determinant J_{45}.*

Combining this with Theorem 4.3.6, we get, from the invariants $J_{r,0}$ and covariants $J_{r-3,1}$, the following sets of invariants of the hexahedral form (i.e., in the coordinates $\overline{a}, \ldots, \overline{f}$):

- $I_{12}', I_{18}', I_{30}', I_{36}', I_{54}'$,
- $I_6'', I_{24}'', I_{30}'', I_{42}'', I_{48}''$.

(4.111)

Consider the following system of equations, linear in the u_a once values for $I_6'', \ldots, I_{48}''$ have been choosen:

$$J_{1,1} = I_6'', \quad J_{7,1} = I_{24}'', \quad \ldots, \quad J_{15,1} = I_{48}''; \tag{4.112}$$

together with the relation $\sum u_a = 0$, the system (4.112) can be *solved* for u_a, up to the same factor λ as occured in the ratios of the x_a. The reduction from the equation problem for $G_{6,2}$ to the Burkhardt form problem is as follows:

1) The given values of $I_6, \ldots, I_{30}$ (the equation problem for $G_{6,2}$) are the values for $I'_{12}, \ldots, I'_{54}$ of (4.111), and these are the values of $J_4, \ldots, J_{18}$ for the Burkhardt form problem.

2) A solution is a point $(x_a, \ldots, x_f) \in \mathbb{P}^4$, a factor of proportionality λ is fixed by the ratio J_6/J_4.

3) The values of $I''_6, \ldots, I''_{48}$ are the values of the linear covariants of the Burkhardt group, determined as in (4.110), and the values of $u_a, \ldots, u_f$ are found as the solution of (4.112), up to the same factor of proportionality λ.

4) From the equations (4.104) we get equations

$$x_a - u_a = 2(\rho - \rho^2)d_2\overline{a}, \ \ldots \ , x_f - u_f = 2(\rho - \rho^2)d_2\overline{f},$$

$$x_a + u_a = 2\overline{\alpha}, \ \ldots \ , x_f + u_f = 2\overline{\zeta}.$$

The first equations determine $\overline{a}, \ldots, \overline{f}$ up to a factor μ, and the second set of equations then determines the value of μ^3 rationally. Moreover, $d_2\mu$ is rationally determined (from (4.17) upon replacing $\overline{a}$ by $\mu\overline{a}$, etc.). But μ need not be determined, as it drops out in the next step.

5) The sought for (x, y, z, t, u) is given by replacing $\overline{a}, \ldots, \overline{f}$ by $\mu\overline{a}, \ldots, \mu\overline{f}$ and ρ by $\mu\rho$ in the following set of equations,

$$\begin{aligned}
6ux &= \rho + 3(-\overline{a} - \overline{b} + \overline{c}),\\
6uy &= \rho + 3(-\overline{d} - \overline{e} + \overline{f}),\\
6uz &= \rho + 3(-\overline{d} + \overline{e} - \overline{f}),\\
6ut &= \rho + 3(\overline{a} + \overline{b} - \overline{c}),\\
6uu &= \frac{(\rho + 3(-\overline{d} + \overline{e} - \overline{f}))(\rho + 3(-\overline{d} - \overline{e} + \overline{f}))}{(\rho + 3(\overline{a} - \overline{b} - \overline{c}))},\\
\rho &= 6\frac{(\overline{b}\overline{c} + \overline{c}\overline{a} + \overline{a}\overline{b} - \overline{e}\overline{f} - \overline{f}\overline{d} - \overline{d}\overline{e}) + d_2}{(\overline{a} + \overline{b} + \overline{c} - \overline{d} - \overline{e} - \overline{f})}.
\end{aligned}$$

These equations are a solution of the form problem for Σ_6.

At this point it is legitimate to ask whether anybody in his right mind is going to start the bewildering calculations involved in the reduction, requiring not only the x_a variables but also the u_a. Probably not, but in this age of computer algebra, it wouldn't be all that difficult. However, the reduction of the problem to the Burkhardt form problem is what we were after, as this will give an alternative solution which will allow us to associate to a given cubic surface ***not only*** a genus 2 curve (as was the case with Klein's solution), but also the local geometry of the Burkhardt quartic at the moduli point. So the equation for the 27 lines has led us naturally to the object of the next chapter, the unique degree 4 invariant of $G_{25,920}$ acting on $\mathbb{P}^4$.