THE FUTURE OF MATHEMATICS

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(Translated by permission from Revue generale des Sciences pures et appliquees, Paris, 19th year, No. 23, December, 1908.)

The true method of forecasting the future of mathematics lies in the study of its history and its present state. And have we not here for us mathematicians, a task in some sort professional? We are accustomed to extrapolation, that process which serves to deduce the future from the past and the present and I 'so well know its limitations that we run no risk of being deluded with its forecasts. In the past there have been prophets incapable of seeing progress, those who have so willingly affirmed that all problems capable of solution have been solved and that nothing remains for future gleaning. Happily the example of the past reassures us. Often enough, already, it has been believed that all problems capable of solution have been solved or at least stated. Then the sense of the word I solution becomes broadened and the insolvable problems become the most interesting of all and undreamed-of problems have arisen. To the Greeks a good solution must employ only the rule and compass; later it became that obtained by the extraction of roots; still later that obtained by the use of algebraic or logarithmic functions. These prophets of no advance thus always outflanked, always I forced to retreat, have, I believe, been forced out of existence. As they are dead I will not combat them. We know that mathematics still develops and our task is to find in what sense. Some one replies, "in every sense;" and in part that is true. But, if absolutely true, it would be somewhat startling. Our riches would soon become an incumbrance and their increase produce an accumulation as incomprehensible as all the unknown truth is to the ignorant. The historian, the physicist himself, must make his selection from among the facts; the brain of the scholar-but a small corner of the universe-could never contain this entire universe; so among the countless facts which nature presents, some must be passed by, others retained. It is as true, a fortiori, in mathematics for neither may the mathematician himself gather pellmell all the facts which come before him. Rather it is he -I was going to say his caprice-which creates them. It is he who constructs from the facts a new combination. Nature does not in general bring this to him ready-made. Doubtless it happens sometimes that the mathematician approaches a problem set by the needs of physics, as when the physicist or the engineer asks of him the calculation of some number in view of an application. Shall we say, we mathematicians, that we must content ourselves to await these commands and, instead of cultivating our science for our pleasure, to have no other care than accommodating ourselves to the tastes of our clients? If there were no other objects for mathematicians than to come to the aid of those who are studying nature it would be from them then that we must await the word of command. Yet is this the right point of view? Certainly not; if we had not cultivated the exact sciences for themselves our mathematical machine would not have been created, and on the day when the word of command came from the physicist we would have been without arms. Nor do the physicists, before studying some phenomenon, wait until some urgent need of life has made the study a necessity, and they are right; had the scientists of the eighteenth century neglected the study of electricity because in their eyes it was but a curiosity of no practical interest we would not have in the twentieth century either the telegraph, or electro-chemistry, or our electrical machinery. The physicist, when forced to choose, is not guided in his selection solely by utility. What brings about then his selection from among the facts of nature? We can not easily say. The phenomena which interest him are those which may lead to the discovery of some law. Those facts interest him which bear some analogy to many other phenomena, which do not appear as isolated facts but closely grouped with others. An isolated fact can be observed by all eyes; by those of the ordinary person as well as of the wise. But it is the true physicist alone who may see the bond which unites several facts among which the relationship is important though obscure. The story of Newton's apple is probably not true, but it is symbolical; so let us think of it as true. Well, we must believe that many before Newton had seen apples fall, but they made no deduction. Facts are sterile until there are minds capable of choosing between them and discerning those which conceal something and recognizing that which is concealed; minds which under the bare fact see the soul of the fact. That is exactly what we do in mathematics; out of the various elements at our disposal we could evolve millions of different combinations, but one of these combinations by itself alone is absolutely void of value. Oftentimes we take much trouble in its construction, but that serves absolutely for naught, unless possibly to give a task for further consideration. But it will be wholly different on the day that that combination takes its place in a class of like results and we have noted this analogy. We are no longer in the presence of a bare fact but of a law. And the true inventor is not the work-man who has patiently built some few of these combinations, but he who has shown their relationships, their parentage. The former saw only the mere fact, the other alone felt the soul of the fact. Oftentimes for the indication of this parentage it has served the inventor's purpose to invent a new name and this name becomes creative; the history of science will supply us with innumerable such instances. The celebrated Viennese philosopher, Mach, states the role of science to be the production of economy of thought just as a machine produces economy of labor. And that is very just. The savage counts with his fingers or with his assemblage of pebbles. By teaching the children the multiplication table we spare them later innumerable countings of pebbles. Someone, sometime, has discovered with his pebbles, or otherwise, that 6 times 7 makes 42; it occurred to him to note the fact and he thus spared us the necessity of doing it over again. He did not waste his time even though his calculation was only for his own pleasure; his operation cost him but two minutes; it would have cost two thousands of millions of minutes had a thousand of million of men to recompute it after he had. The importance of a fact is known by its fruits, that is to say, by the amount of thought which it enables us to economize. In physics, the facts of great fruitage are those which combine into some very general law, because they then allow us to predict a great number of other facts, and it is just the same with mathematics. I have devoted myself to a complicated calculation and have come laboriously to a result; but I will not feel repaid for my pains if I am not now able to foresee the results of other analogous calculations and to pursue such calculations with sure steps, avoiding the hesitations, the gropings of the first time. I shall not have wasted my time, on the contrary, if these gropings have ended in revealing to me in the problem which I have just treated some hidden relationship with a far more extended class of problems. If at the same time they have shown me resemblances and differences; if, in short, they have made me foresee the possibility of a generalization, then it is not merely a new answer which I have acquired; it is a new force. An example which comes at once to mind is the algebraic formula which gives us the solution of a class of numerical problems when its letters are replaced by numbers. Thanks to the formula, a single algebraic demonstration spares us the pains of going over the same ground time after time for each new calculation. But this gives us only a very rough illustration. Everyone knows that there are analogies, some most valuable, which can not be expressed by a formula. If a new result has value it is when, by binding together long- known elements, until now scattered and appearing unrelated to each other, it suddenly brings order where there reigned apparent disorder. It then allows us to see at a glance the place which each one of these elements occupies in the ensemble. This new fact is not alone important in itself, but it brings value to all the older facts which it now binds together. The brain is as weak as the senses, and it would be lost in the complexities of the world were there not harmony in that complexity. After the manner of the short-sighted, we would see only detail after detail, losing sight of each detail before the examination of another, unable to bind them together. Those facts alone are worthy of our attention which bring order into this complexity and so render it comprehensible. Mathematicians attach great importance to the elegance of their methods and results; nor is this pure dilettanteism. Indeed, what brings to us this feeling of elegance in a solution or demonstration? It is the harmony among the various parts, their happy balancing, their symmetry; it is, in short, all that puts order among them, all that brings unity to them and which consequently gives us a certain command over them, a comprehension at the same time both of the whole and of the parts. But as truly it is that which brings, with it a further harvest, for, in fact, the more clearly we comprehend this assemblage, and at a glance, the better we will realize its relationships with neighboring groups, the greater consequently will be our chances of divining further possible generalizations. Elegance may arise from the feeling of surprise in the unexpected association of objects which we had not been accustomed to group together; It occurs frequently from the contrast between the simplicity of the means employed and the complexity of the given problem; we consequently reflect as to the reason of this contrast and almost without fail we find the cause not in pure hazard, but in some unexpected law. In a word; the sentiment of mathematical elegance is naught else than the satisfaction due to some, I know not just what, adaptation between the solution just found and the needs of our mind, and it is because of this adaptation itself that the solution becomes an instrument to us. This aesthetic satisfaction is therefore connected with the economy of thought. Thus the caryatides of the Erechtheum engender in us the same feeling of elegance, for example, because they carry their heavy load with such grace, or we might say so cheerfully, that they produce in us a feeling of economy of effort. It is for the same reason that when a somewhat long calculation, has led us to a simple and striking result we are not fully satisfied until we have shown that we could have foreseen, if not the whole result, at least its most characteristic details. Why? What is it that prevents our satisfaction with this accomplished calculation giving all which we seemed to desire? It is because our long calculation would not again serve in another analogous case and because we have not used that mode of reasoning, often half intuitive, which would have allowed us to foresee our result. When our process is short we may see at a glance all its steps, so that we may easily change and adapt it to whatever problem of the same nature may occur, and then, since it allows us to foresee whether the solution of the problem will be simple, we can tell at least whether the problem is worth undertaking. What we have just said suffices to show how vain would be any attempt whatever to replace by any mechanical process the free initiative of the mathematician. To obtain a result of real worth it will not suffice to grind it out or to have a machine for putting our facts in order. It is not alone order but the unexpected order which is I of real worth. The machine may grind upon the mere fact, but the soul of the fact will always escape it. Since the middle of the last century mathematicians have been more and more anxious for the attainment of absolute rigor in their processes; they are right, and that tendency will increase more and more. In mathematics rigor is not everything, but without it there would be nothing; a demonstration which is not rigorous is void. I believe no one will contest this truth. But to take this too literally would bring the conclusion, for example, that before 1820 there was no mathematics. That is surely going too far; then the geometricians assumed willingly what we explain by a prolix discussion. This does not mean that they did not realize their omission, but they passed it over too rapidly, and for greater surety they would have had to go through the trouble of giving this discussion. But is it necessary to repeat every time this discussion? Those who, first in the field, had to be preoccupied with all this rigor have given to us demonstrations which we could try to imitate; but if the demonstrations of the future must be built upon this model our mathematical treatises would become too long, and if I fear this length it is not only because I dread the incumbrance of our libraries, but also because I fear that in this lengthening of our demonstrations they will lose that appearance of harmony of which I have just shown the so serviceable role. We should always aim toward the economy of thought. It is not enough to give models for imitation. It must be possible to pass beyond these models and, in place of repeating their reasoning at length each time, to sum this in a few words. And this has now and then been already accomplished; for instance, there was a whole type of demonstrations which were perfectly similar and repeatedly occurring; they were perfectly rigorous, but tedious; one day some one thought of applying the word convergence and that word has taken their place. There is now no need of repeating these processes, for they are understood. Those who have cut our difficulties in quarter have rendered us double service-first, they have taught us to do as they have done when there is need, but above all to avoid this process as often as we can without the loss of this rigor. We have just seen, through an example, the importance of words in mathematics, but I could cite many more cases. It is scarcely credible, as Mach said, how much a well-chosen word can economize thought. I do not know whether or not I have said somewhere that mathematics is the art of giving the same name to different things. We must so understand it. It is meet that things different in substance but like in form should be run in the same mold, so to speak. When our language is well chosen it is astonishing to see how all the demonstrations made upon some known fact immediately become applicable to many new facts. Nothing has to be changed, not even the words, since the names are the same in the new cases. There is an example which comes at once to my mind; it is quaternions, upon which, however, I will not dwell. A word well chosen very often causes the disappearance of exceptions to rules as announced in their former forms; it was for this purpose that the terms negative quantities, imaginary quantities, infinite points, have been invented. And let us not forget that these exceptions are pernicious, for they conceal laws. Very well then, one of those marks by which we recognize the pregnancy of a result is in that it permits a happy innovation in our language. The mere fact is oftentimes without interest; it has been noted- many times, but has rendered no service to science; it becomes of value only on that day when some happily advised thinker perceives a relationship which he indicates and symbolizes by a word. The physicists also do just the same way. They invented the term energy, a word of very great fertility, because through the elimination of exceptions it established a law; because it gave the same name to things different material but similar form. Among the words which have had this happy result I will mention the group and the invariant. They make us perceive the gist of many mathematical demonstrations; they make us realize how often mathematicians of the past must have run across groups without recognizing them and how, believing these groups such isolated things, they have found them in close relationship without knowing why. To-day we would say that they were looking right in the face of isomorphic groups. We feel now that in a group the substance interests us but very little; it is the form alone which matters, and so, when we once know well a single group, then we know through it all the isomorphic groups; thanks to the words groups and isomorphism, which sum in a few syllables this subtle law and make it at once familiar to us all, we take our step at once and in so doing economize all effort of thought. The idea of group, moreover, is bound up with that of transformation. Why then do we attach so much value to the invention of a new transformation? Because from a single theorem we may deduce ten or twenty; it has a value similar to the addition of a zero at the right of an integral number. We now realize what has determined the direction of the advance of mathematics in the past and the present and it is as certain what will determine it in the future. But the nature of the problems which come up will contribute equally. We must not forget what should be our goal; according to me that end is double. Our science confines itself at the same time to philosophy and to physics, and it is for these two neighbors that we work. And so we have always seen and always will see mathematics progressing in two opposite directions. In one sense mathematics must return upon itself and that is useful, for in returning upon itself it goes back to the study of the human mind which has created it rather than to those creations which borrow the least bit from the external world. That is why certain mathematical speculations are useful, such as those whose aim is the study of postulates, of unusual geometries, of functions having peculiar values. The more these speculations depart from our common conceptions and consequently from nature or practical applications, the better they show us the working of the human mind which constructs them when it becomes freed from the tyranny of the external world, and the better, in consequence, it comes to know itself. But it is to the opposite side-the side of nature-against which we must direct the main corps of our army. There we meet the physicist or the engineer who says to us: "Can you integrate for me such a differential equation? I must have it within eight days because of a certain construction which must be finished by that time." "That equation," we reply, "is not of an integrable type; you know there are many like it." " Yes, I know that; but of what use are you then?" More often, however, there is a better understanding. The engineer does not need his integral in finite terms. He needs only a rough value of the integral function, or perhaps only a certain numerical result which he could easily deduce from such a value of the integral if he had it. Ordinarily we could get this numerical result for him if we knew just how accurate it must be-that is, with what approximation. Formerly an equation was not considered solved except when the solution was expressed by means of a finite number of known functions; but that is possible scarcely once in a hundred times. What we can always do, or rather what we may always try to do, is to solve the problem qualitatively, so to speak-that is, to find the general shape of the curve which the unknown function represents. It remains, then, to find the quantitative solution of the problem; but if the unknown can not be determined as a finite result it can f always be represented by means of an infinite convergent series which will allow the numerical calculation. May we regard this as a true solution? It is related that Newton once communicated to Leibnitz an anagram something like this:

aaaaabbbeeeeii, etc.

Leibnitz naturally was wholly at a loss as to its meaning; but we who have the key know the signification of that anagram and translating it into ordinary language it becomes: I know how to integrate all differential equations; and we are led to say to ourselves that Newton had strange good luck with such a singular illusion. He would have said all simply, that he could form (by the method of undetermined coefficients) a series of powers satisfying formally the given equation. Such an apparent solution would no longer satisfy us to-day; and that for two reasons, because its convergence would be too slow and because the terms would follow one another according to no definable law. On the other hand, the series θ seems to us to leave nothing to be desired, first, because it converges very rapidly (and that because the engineer wishes his result as quickly as possible), and then because we may see at a glance the law of its terms (that, for the satisfaction of the esthetic needs of the mathematician). But there are no longer some problems which are solved and others which are not; there are only problems more or less solved accordingly as they are represented by a series converging more or less rapidly and following a law more or less harmonious. It occurs sometimes that an imperfect solution leads to a better one. Sometimes the series converges so slowly that calculations from it are impracticable, and we have shown only the possibility of a solution. And then the engineer thinks the solution only derisory, and he is right, as it will not allow I him to finish his construction on the given date. He cares little I whether the solution will be useful to the engineer of the twenty- second century; we feel otherwise, and are sometimes as happy if we have saved for our grandson as for our contemporaries. Sometimes, trying this way and that, empirically, we might say, we happen upon a formula sufficiently convergent. "What more do you want?" we ask the engineer; and yet, despite that, we are not satisfied ourselves." Why? Could we have foreseen it the first time, we might a second. We have reached a solution; that is a small matter to us if we have no sure hope of getting it a second time. As a science grows it becomes more and more difficult to know it all. Then we cut it up into bits and each one contents himself with a bit; in a word, we specialize. If this process continues it will become a vexatious obstacle to the progress of our science. We have said that it is the unexpected bringing together of diverse parts of our science which brings progress. Too much specialization prevents this. Let us hope that a congress like this, bringing us into closer relationships with each other and spreading before the eyes of each his neighbor's fields, obliging us to compare these fields, so that we set forth for awhile from our own little villages, will annul this danger to which I have just called attention. But I have stopped too long over generalities. Let us pass in review the diverse parts which form the whole science of mathematics, let us see what each branch has done, whither each tends and what we may hope from each. If the views we have just expressed are right, the great advances of the past will be found where two of these branches have approached each other, where the similarity of their forms despite the dissimilarity of material has become evident, where one has been modeled upon the other in such manner that each takes profit from the other. At the same time we should foresee the progress of the future in interlockings of the same nature.

I. ARITHMETIC.

The progress of arithmetic has been slower than that of algebra or analytical geometry, and the reason is very evident. Arithmetic does not present to us that feeling of continuity which is such a precious guide; each whole number is separate from the next of its kind and has in a sense individuality; each in a manner is an exception and that is why general theorems are rare in the theory of numbers; and that is why those theorems which may exist are more hidden and longer escape those who are searching for them. But if arithmetic is less developed than algebra and analytical geometry it may well model itself upon those branches and take profit by their advances. The arithmetician must take for his guide the analogies with algebra. These analogies are many, and if often they have not so far proved very useful yet they have at least been known for some time; the language itself of the two branches shows this; for instance when we speak of transcendental numbers and when we take into account that the future classification of these numbers images that of transcendental functions; still it is difficult to see how we can pass from one classification to the other; however, the step has already been taken, so it is no longer the task of the future. The first example which comes to mind is the theory of congruents where we find a perfect parallelism with that of algebraic equations. And we will certainly complete this parallelism which must exist between the theory of algebraic curves and that of congruents of two variables, for instance. And when the problems relative to congruents of several variables are solved we shall have taken the first step toward the solution of many of the questions of indeterminate analysis. Another example where the analogy has not always been seen at first sight is given to us by the theory of corpora and ideals. For a counterpart let us consider the curves traced upon a surface; to the existing numbers correspond the complete intersections to the ideals the incomplete intersections, and to the prime ideals the indecompostable curves; the various classes of ideals thus have their analogs. There can be no doubt that this analogy can throw light upon the theory of ideals, or upon that of surfaces, or perhaps on both at the same time. The theory of forms, and in particular that of quadratic forms, is intimately bound with that of ideals. Among the theories of arithmetic this was one of the first to take shape and it came when the arithmeticians introduced unity through the considerations of groups of linear transformations. These transformations permitted classification and consequently the introduction of order. Perhaps we have obtained all the fruit which could be hoped for; but if these linear transformations are the 'parents of geometrical perspectives, analytical geometry may furnish many other transformations (as, for example, the birational transformations of an algebraic curve) for which it may be well worth our while to look for arithmetical analogs. Doubtless these will form discontinuous groups of which we must first study the fundamental parts as the key to the whole. I have no doubt that in this study we will make use of Minkowski's Geometrie der Zahlen (Geometry of Numbers). An idea from which we have not yet taken all that is possible is the introduction by Hermite of continuous variables in the theory of numbers. Let us start with two forms F and F', the second quadratic determinate, and apply to both the same transformation; if the. form F' transformed is reduced, we will say that the transformation is reduced and also that the form F transformed is reduced. It then follows that if the form F can be transformed to itself it can have many reductions; but this inconvenience is essential and can be avoided by no subterfuge. On the other hand these reductions do not prevent a classification of the forms. It is clear that this idea which has hitherto been applied only to limited classes of forms and transformations can be extended to groups of nonlinear transformations and we may yet hope to have a harvest greater than has ever been reaped from it. An arithmetical domain where unity seems absolutely absent is found in the theory of prime numbers; the laws of asymptotes have been found and we must not hope for others; but these laws are isolated and are reached only by different paths which seem to have no intercommunication. I believe that I have a glimpse of the wished for unity, but I see it only vaguely; all leads back without doubt to the study of a family of transcendental functions which, through the study of singular points and the application of the method of M. Darboux, will permit the calculation asymptotically of certain functions of very great numbers.

II. ALGEBRA.

The theory of algebraic equations will still hold for a long while the attention of geometricians; the sides from which it may be approached are numerous and diverse; the most important is that of the theory of groups, to which we will return. But there is also the question of the calculation of the numerical value of roots and the discussion of the number of real roots. Laguerre has shown that not all was said upon this point by Sturm. Then there is the study of the system of invariants which do not change sign when the number of real roots remains the same: We may also form series of powers representing functions which may have for singular points the various roots of an algebraic equation (for instance, rational functions of which the denominator is the first member of this equation); the coefficients of the terms of high order will furnish one of the roots with an approximation more or less close; there is here the germ of a process of numerical calculation to which a systematic study could be given. During a period of forty years the study of invariants of algebraic forms seems to have absorbed all algebra; they are to-day laid aside, although the subject has not been exhausted; but we must no longer limit the study to the invariants of linear transformations; it is to be extended to those referring to any group whatever. The theorems, acquired in the past have suggested others more general which are grouping about them much as a crystal grows from a solution. And, as to the theorem of Gordan that the number of distinct invariants is limited, the demonstration of which Hilbert has so happily simplified, it seems to me that it leads to a problem much more general: If we have an infinity of whole polynomials, depending algebraically from a finite number among them, can we always deduce them from a finite number among them by addition and multiplication? We must not believe that the task of algebra is finished because we have found rules for all the possible combinations. We have still to search out the interesting combinations, those which satisfy such and such conditions. Thus there will be established a sort of indeterminate analysis in which the unknowns will not be whole numbers but polynomials. Then in this case algebra will model itself upon arithmetic and take as a guide the analogy of the whole number, either as a whole polynomial of any coefficients whatever or as a whole polynomial of whole coefficients.

III. DIFFERENTIAL EQUATIONS.

Much has already been done for linear differential equations and it remains to perfect what has been commenced. But with nonlinear differential equations there has been much less advance. The hope of an integration by the aid of known functions has been given up long since; therefore we must study for themselves the functions defined by these differential equations and then attempt a systematic classification; the study of the mode of change in the neighborhood of singular points doubtless will furnish the first elements of such a classification, but we will be satisfied only when we shall have found a group of transformations (for instance, the transformations of Cremona) which will play with respect to the differential equations the same role as the group of birational transformations does for the algebraic equation. We can then group in the same class all the transformations of the same equation. We shall have for our guide the analogy with a theory already made-that of birational transformations and the genus of an algebraic curve. We may propose to lead back the study of these functions to that of uniform functions, and this in two ways: We know that if $y = f(x)$, we can, whatever may be the $f(x)$, express y and x by uniform functions of an auxiliary variable t; but, if $f(x)$ is the solution of a differential equation, in what case will the uniform auxiliary functions themselves satisfy the differential equation? We do not know; neither do we know in what cases the general integral can be put in the form $F(x, y)$ = arbitrary constant, where $F(x, y)$ is a uniform function. I will urge the qualitative discussion of the curves defined by differential equations. In the simplest case, that in which the equation is of the first order and the first degree, this discussion leads to the determination of the number of limited cycles. It is very sensitive and what will help us is the analogy with the method of the determination of the number of real roots of an algebraic equation; when-ever any step whatever shows the real status of this analogy we may be sure of a very great advance.

IV. EQUATIONS WITH PARTIAL DERIVATIVES.

Our knowledge of equations containing partial derivatives has taken recently a very considerable step in advance by means of the discoveries of M. Fredholm. If we examine closely the basis of these discoveries we will find that this difficult theory is modeled upon an-other more simple, that of determinants and of systems of the first degree. In the greater part of the problems of mathematical physics the equations to be integrated are linear; they serve to determine unknown functions of several variables, functions which are continuous. Why? Because we have made the equations in conformity with the supposition that matter is continuous. But matter is not continuous; it is formed of atoms; had we wished to write equations as they should be for an observer whose sight is sufficiently keen to see these atoms, we would not have had a small number of differential equations serving to determine certain unknown functions, we would have had a very great number of algebraic equations for determining a great number of unknown constants. And these algebraic equations would have been linear and of such a nature that with infinite patience we could have applied directly to them the methods of determinants. But, since the brevity of our lives will not allow us this luxury of infinite patience, we must proceed otherwise; we must pass to the limit and suppose matter continuous. There are two ways of generalizing the theory of equations of the first degree in passing to the limit. We can consider an infinity of separate equations with an infinity, equally independent of unknowns. This has been done, for example, by Hill in his theory of the moon. We will then have infinite determinants which are to ordinary determinants as series are to finite sums. We can take an equation of partial derivatives representing, we may say, a continuous infinity of equations, and use them to determine an unknown function representing a continuous infinity of unknowns. We then have other infinite determinants which are to ordinary determinants as integrals are to finite sums. Fredholm used this method; his success moreover came from his utilization of the following fact: If, in a determinant, the elements of the principal diagonal are equal to unity and the other elements are homogeneous and of the first order, we can arrange the development of the determinant by combining in a single group all the homogeneous terms of the same degree. The infinite determinant of Fredholm may be so arranged and it happens that we thus obtain a converging series. Has this analogy which certainly guided Fredholm given us all it ought to? Certainly not. If his success came from the linear form of the equations we should be able to apply ideas of the same nature to all problems having equations of linear form, and, indeed, to ordinary differential equations, since their integration may be always reduced to that of linear equations of partial derivatives of the first order. Recently the problem of Dirichlet and those connected with it have been approached by another method, returning to the original one of Dirichlet and searching for the minimum of a definite integral except that this is now done by rigorous processes. I do not doubt that these two methods without much difficulty will be made comparable and advantage taken of their mutual relationships. Nor do I doubt that both will have much to gain by such a comparison. Thanks to M. Hilbert, who has been doubly an initiator, we are already on that path.

V. THE ABELIAN FUNCTIONS.

The principal question remaining to us for solution concerning Abelian functions we know. The Abelian functions begot by the integrals relative to an algebraic curve are not the most general ones; they belong only to a particular case, so we may call them special Abelian functions. What is their relationship to the general functions and how shall we classify these latter? But a short time ago the solution of these problems seemed far distant. I believe that it is virtually solved to-day, now that MM. Castelnuovo and Enriques have published their recent memoir upon the integrals of total differentials of the varieties of more than two dimensions. We know now that there are Abelian functions belonging to a curve and others to a surface, and that it will never be necessary to extend them to more than two dimensions. Combining this result with what we may obtain from the works of M. Wirtinger we will doubtless reach the end of all our difficulties.

VI. THE THEORY OF FUNCTIONS.

It is especially with regard to functions of two and of several variables that I wish to speak. The analogy with the functions of a single variable gives a valuable but insufficient guide; there is an essential difference between the two classes of functions, and every time a generalization is attempted by passing from one to the other an unexpected obstacle has been encountered which has sometimes been overcome by special artifices, but which so far has more often remained insurmountable. We must therefore search for facts from first principles to make clear to us this difference between functions of one variable and those containing several. We should look first more closely at the devices which have brought success in certain cases to see what they may have in common. Why is a conformal representation more often impossible in the domain of four dimensions and what shall we substitute for it? Does not the true generalization of functions of one variable come in the harmonic functions of four variables of which the real parts of the functions of two variables are only particular cases? Can we make use of what we know of algebraic or rational functions in the study of transcendental functions of several variables? Or, in other words, in what sense may we say that the transcendental functions of two variables are to transcendental functions of one variable as rational functions of two variables are to rational functions of one variable? It is true that if $z = f(x, y)$ we can, whatever the function f may be, express x, y, z, respectively, as uniform functions of two auxiliary variables, or, to employ an expression which has become common for this process, can we make uniform the functions of two variables as we do those of one? I limit myself to the setting of the problem, the solution of which may perhaps come in the future.

VII. THE THEORY OF GROUPS.

The theory of groups is an extensive subject upon which there is much to be said. There are many kinds of groups, and whatever classification may be adopted we will always find new groups which will not fit it. I wish to limit myself and will speak here only of the continuous groups of Lie and the discontinuous ones of Galois, both of which we are now wont to classify as groups of finite order, although the term does not apply to both groups in the same sense. In the theory of the groups of Lie we are guided by a special analogy; a finite transformation is the result of the combination of an infinity of infinitessimal transformations. The simplest case is that where the infinitessimal transformation is equivalent to the multiplication by $1+\varepsilon$, where ε is very small. The repetition of these transformations gives rise to the exponential function; that was Neper's method of procedure. We know that an exponential function can be expressed by a very simple and very convergent series, and analogy should then show us what path to follow. Moreover, that analogy may be expressed by a special symbolism upon which you will excuse me from dwelling. We are already well advanced along this path, thanks to Lie, Killing, and Cartan; it remains only to simplify the demonstrations and to coordinate and classify the results. The study of the groups of Galois is much less advanced, and for a very simple reason, that same reason which makes arithmetic behindhand to analytical geometry, that lack of continuity which is of such great use for our advances. But happily there is a manifest parallelism between the two theories and we must try to put this more and more in evidence. This analogy is exactly parallel to that which we have noted between arithmetic and algebra and we should derive from it similar aid.

VIII. GEOMETRY.

It seems at first sight as if geometry could contain nothing which is not already presented to us in algebra and analytical geometry; for the facts of geometry are nought else than the facts of algebra and analytical geometry expressed in another language. One might think then, after the review which we have just made, that there would remain nothing further to say specially about geometry. But we would then be unmindful of a well-built language, mode of argument, of something which adds to the things themselves a mode of expressing them and consequently of grouping them. And, moreover, geometrical considerations lead us to propose new problems; they are, indeed, if you so choose to call them, analytical problems, but they would never have been proposed through analytical geometry alone. Meanwhile analytical geometry profits from these just as it has profited from the problems it has been called upon to solve for physics. Common geometry has a great advantage in that the senses may come to the help of our reason and aid it in finding what path to follow, and many minds prefer to put their problems of analytical geometry in the ordinary geometrical form. Unfortunately our senses can not lead us so very far, and they fail us when we try to escape from the classical three dimensions. Must we say that, departing from the limited domain where our senses seem to wish to confine us, we must no longer count upon pure analysis and that all geometry of more than three dimensions is vain and useless? In the generation which preceded us the greatest masters would have replied "yes." We have nowadays become so familiar with this notion of more than three dimensional space that we may speak of it even in the university without arousing astonishment. But what purpose can geometry serve? It gives us, close at hand, a most convenient language which can express very concisely what the language of analytical geometry can express only in very prolix phraseology. Moreover, its language gives the same name where there are resemblances and affirms analogies so that we do not forget them. And even more, it guides us into that space which is too vast for us and which we may not see; it does this by ever bringing to mind the relationship of the latter space to our ordinary, visible space, which without doubt is only a very imperfect image, but which nevertheless is an image. Here further, as in all the preceding instances, this analogy with what is simple allows us to comprehend that which is complex. This geometry of more than three dimensions is not a simple analytical geometry; it is not purely quantitative; it is also qualitative, and it is in the latter sense that it becomes especially interesting. The importance of the Analysis Situs is very great; I can not insist too much on that; the advance which it has taken from Riemann, one of its chief creators, is enough to indicate this. It is essential that it should be constructed completely in hyperspace. We would be then furnished with a new sense, one capable of seeing really into hyperspace. The problems of the Analysis Situs would perhaps not have been thought of had there been only the language of analytical geometry; or rather, I am wrong, they would certainly have been set, since their solution is necessary for many of the questions of analytical geometry; but they would have been set one after another with no indication of a common bond between them. It is the introduction of the ideas of transformations and groups which has contributed especially to the recent progress in geometry. We owe to these that geometry is no longer an assemblage of more or less curious theorems which follow each other with no resemblances; they have now acquired a unity; and, furthermore, we must not forget in our history of science that it was for the sake of geometry that a systematic study was started of continuous transformations, so that pure geometry has contributed its part to the development of the idea of the group so useful in the other branches of mathematics. The study of groups of points upon an algebraic curve, according to the method of Brill and Noether, has given us also fruitful results either directly or as serving as models for analogous theories. We have thus seen develop a whole chapter of geometry where the curves traced upon a surface playa role similar to that of a group of points upon a curve. And from this very day on, we may hope to see in this way light thrown on the last mysteries which exist in the study of surfaces and which have been so difficult to solve. The geometricians have thus a vast field from which to reap a harvest. I must not forget enumerative geometry, and especially infinitesimal geometry, cultivated with such brilliancy by M. Darboux, and to which M. Bianchi has added such useful contributions. If I do not say more upon this subject it is because I have nothing to add after the brilliant lecture by M. Darboux. 1.

IX. CANTORISM.

I have already spoken of the need we have of continually going back to the first principles of our science and the profit we may thus obtain in the study of the human mind. It is this need which has inspired two attempts which hold an important place in the more recent part of mathematical history. The first is Cantorism, whose services to science we will know. One of the characteristic traits of Cantorism is that in place of generalizing and building theorems more and more complicated on top of each other and defining by means of these constructions themselves, it starts out from the genus supremum and defines, as the scholastics would have said, per genus proximum et differentiam specificam. What horror would have been brought to certain minds-that of Hermite, for instance, whose favorite idea was comparing the mathematical to the natural sciences! With the most of us these prejudices have passed away, but it still happens that we come across certain paradoxes, certain apparent contradictions which would have overwhelmed Zenon d'Elee and the school of Megore with joy. I think, and I am not the only one who does, that it is important never to introduce any conception which may not be completely defined by a finite number of words. Whatever may be the remedy adopted, we can promise ourselves the joy of the physician called in to follow a beautiful pathological case.

X. THE RESEARCH OF POSTULATES.

And yet, further, we are trying to enumerate the axioms and postulates, more or less deceiving, which serve as the foundation stones of our various mathematical theories. M. Hilbert has obtained the most brilliant results. It seems now as if this domain must be very limited and that there will not be any more to be done when this inventory is finished, and that will be very soon. But when all has been gathered together there will be plenty of ways of classifying them, and a good librarian will always find something to busy himself with and each classification will be instructive to the philosopher. I stop this review, which I could not hope to make complete, for many reasons, and because I have already drawn too much on your patience. I believe that my examples will have been sufficient to show you, by what means the mathematical sciences have progressed in the past and along what paths they must proceed in the future.

1. See G. Darboux: Les origines, les methodes et les problemes de la Geometrie infinitesimale (The origin, methods, and problems of infinitesimal geometry). Revue generale des Sciences, 15 Nov., 1908.

Address delivered April 10, 1908, at the general session of the Fourth International Congress of Mathematicians (Rome, April 6-11, 1908); previously published in pamphlet form by and at the expense of the Mathematical Society of Palermo. M. Poincare was unable to deliver this lecture and M. Darboux graciously undertook the task. To M. Guccia we express our gratitude for the authority which he has courteously extended for its reproduction.