Schwarz-Christoffel transformations

It is helpful to have available systematic ways to find a variety useful conformal mappings. Our text treats one such method: bilinear transformations. Here we study a second: Schwarz-Christoffel transformations. These are discussed in many references; I recommend particularly [1], a lovely book on complex variable theory which has a decided applied slant.

The Schwarz-Christoffel transformations solve the following problem: we are given a certain polygon lying in the $w$-plane, and we want to find a conformal mapping $w = f(z)$ which carries the real axis onto the boundary of the polygon and the upper half plane $D$ into the interior $D'$ of the polygon. Figure 1 shows the basic set-up, assuming that the polygon has five vertices:

![Figure 1: Basic set-up for the Schwarz-Christoffel transformations.]

Here are a few more details of the general setup:

- If the polygon has $n$ vertices then we will denote these by $w_1, w_2, \ldots, w_n$. As shown in the figure, these are numbered in order counterclockwise around the polygon. We assume that these $n$ points are given and fixed.

- Correspondingly, there will be $n$ points $z_1, \ldots, z_n$ on the real $z$ axis, numbered from left to right, such that $f(z_1) = w_1, f(z_2) = w_2$, etc. It turns out that we can decide arbitrarily where three of these points will be located; then determining where the remaining $z_i$ must be put is part of the problem. However, considerations of symmetry can often help find these, as we shall see.

- It is frequently convenient to choose one of the points $z_i$ to lie at infinity; when we do this we will usually let this point be $z_n$.

In practice (for example, if we wish to solve Laplace’s equation in the region $D'$) we may want a mapping the other way—from $D'$ to the upper half plane. Of course, this is given by the inverse of the function $f(z)$ and may be found, at least in principle, by writing $w = f(z)$ and solving for $z(w)$.

To write down a formula for the Schwarz-Christoffel transformation we will need to introduce the external angles at the corners of the polygon, and we will write these as (fractional) multiples of $\pi$, letting $\gamma_i \pi$ be the external angle at vertex $w_i$. This is defined as follows: if we travel from $w_{i-1}$ to $w_i$ along the edge of the polygon which joins these vertices, then $\gamma_i \pi$ is the angle through which we must turn, counterclockwise, to start traveling toward $w_{i+1}$. Because in traversing the whole polygon in this way we must turn a total of $360^\circ$, these angles satisfy $\sum_{i=1}^{n} \gamma_i = 2$. See Figure 2. Note that in this figure $\gamma_3$ is negative, since we must turn clockwise as we pass $w_3$. 

1
The key to obtaining $f$ is to recall that a general fact which we discussed in class in the process of discussing the definition of conformal mapping:

If $z(t)$ is a curve in the $z$ plane then the direction of that curve at the point $z_0 = z(t_0)$ is the tangent vector $\dot{z}(t_0)$. If $w = f(z)$ is a conformal map, then the direction of the image curve $w(t) = f(z(t))$ at $w_0 = f(z_0)$ is $\dot{w}(t) = f'(z_0)\dot{z}(t_0)$, that is, the map rotates the direction of the curve counterclockwise by an amount $\arg f'(z_0)$.

Now the Schwarz-Christoffel transformation $w = f(z)$ which we obtain will in fact be analytic not only in the upper half plane but also at all points of the real $z$ axis except the special points $z_1, z_2, \ldots, z_n$. (Technically what this means is that it will be analytic in a plane with $n$ cuts, one starting at each point $z_i$ and going directly down to $\infty$.) Consider a curve $z(t) = z_i + t(z_{i+1} - z_i)$, $0 < t < 1$, which traces out just the piece of the real axis lying between $z_i$ and $z_{i+1}$, moving at constant speed; we want $f$ to send this curve to the straight line from $w_i$ to $w_{i+1}$. But $\dot{z}(t) = z_{i+1} - z_i$ is constant (it is a positive real number). We want $\arg \dot{w}(t)$ to be constant, since $w(t)$ is moving along a straight line, and since $\dot{w}(t) = f'(z(t))\dot{z}(t)$ we conclude that

$$\arg f'(z) \text{ must be constant on the real axis between } z_i \text{ and } z_{i+1}. \quad (1)$$

Now from the above it follows that the direction of the side of the polygon between between $w_i$ and $w_{i+1}$ is $f'(z)$, where $z$ is any point on the real axis between $z_i$ and $z_{i+1}$. Consideration of Figure 2 then tells us that the argument of $f'(z)$ must increase by $\gamma_i \pi$ as $z$ increases past $z_i$ along the real axis past $z_i$, that is,

$$\text{if } z_{i-1} < z < z_i \text{ and } z_i < \tilde{z} < z_{i+1} \text{ then } \arg f'(\tilde{z}) = \arg f'(z) + \gamma_i \pi. \quad (2)$$

The conditions (1) and (2) essentially determine $f'(z)$.

One function which satisfies (1) and (2) is

$$f'(z) = A(z - z_1)^{-\gamma_1}(z - z_2)^{-\gamma_2} \cdots (z - z_n)^{-\gamma_n} \quad (3)$$

Here we assume that $f'(z)$ is defined is defined in the plane with cuts as described above—that is, a cut from each $z_i$ running downward, parallel to the negative imaginary axis. In this cut plane the factors in (3) are defined by

$$(z - z_i)^{-\gamma_i} = \exp[-\gamma_i \log(z - z_i)] = \exp[-\gamma_i (\log|z - z_i| + i \arg(z - z_i))],$$
with $-\pi/2 < \arg(z - z_i) < 3\pi/2$. Thus for $z$ real we use $\arg(z - z_i) = 0$ for $z > z_i$ and $\arg(z - z_i) = \pi$ for $z < z_i$, so that for $z_i < z < z_{i+1}$,

$$\arg f'(z) = \arg A - \sum_{k=1}^{n} \gamma_k \arg(z - z_k) = \arg A - \pi \sum_{k=i+1}^{n} \gamma_k,$$  \hspace{1cm} (4)$$

from which (1) and (2) follow immediately. Integrating (3) yields

$$f(z) = A \int z (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_n)^{-\gamma_n} d\zeta + B,$$  \hspace{1cm} (5)$$

and this is the formula for the Schwarz-Christoffel transformation that we have been seeking. The constants $A$ and $B$, which must be determined, have a simple geometric interpretation: the integral in (5) produces a certain polygonal image of the upper half plane; the multiplicative factor $A$ scales the image by $|A|$ and rotates it by $\arg A$, and the additive constant $B$ shifts the image by that amount.

As indicated above, we may also choose the points $z_i$ so that $z_n = \infty$. In that case, the formulas (3) and (5) become

$$f'(z) = A(z - z_1)^{-\gamma_1} (z - z_2)^{-\gamma_2} \cdots (z - z_{n-1})^{-\gamma_{n-1}}$$  \hspace{1cm} (6)$$

and

$$f(z) = A \int z (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_{n-1})^{-\gamma_{n-1}} d\zeta + B$$  \hspace{1cm} (7)$$

These may be derived along the same lines. We give further discussion of both (5) and (7) in the Appendix to these notes, but it is not necessary to read that in order to learn how to use the Schwarz-Christoffel transformation.

Remark 1: Since we have not specified a lower limit of integration in (5) and (7), the integrals there just denote any antiderivative of the integrand; in this sense $B$ is just an integration constant. If $z_*$ is a point in the $z$-plane which lies either in the upper half plane or on the real axis, and we know that we want $z_*$ to be mapped to a point $w_*$ by $f(z)$, then this determines $B$ and we can rewrite (5) as

$$f(z) = A \int_{z_*}^{z} (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_n)^{-\gamma_n} d\zeta + w_*.$$  \hspace{1cm} (8)$$

We now give several examples of the use of the Schwarz-Christoffel transformation in concrete situations.

Example 1: As a first example, we suppose that the polygon in question is a rectangle; to be definite we suppose that the vertices are $w_3 = a$, $w_4 = a + ib$, $w_1 = -a + ib$, and $w_2 = -a$, where $a$ and $b$ are positive real constants. The external angles then are all $\pi/2$, so that $\gamma_i = 1/2$ for $i = 1, 2, 3, 4$. Guided by considerations of symmetry we will take $z_1 = -t$, $z_2 = -1$, $z_3 = 1$, and $z_4 = t$ for some $t > 1$; we cannot choose $t$ in advance.
but rather must adjust it to give the correct ratio $b/a$ of the sides of the rectangle. See Figure 3.

Now from (3) we have

$$f'(z) = \frac{A}{\sqrt{(z + t)(z + 1)(z - 1)(z - t)}} = \frac{A}{\sqrt{(z^2 - t^2)(z^2 - 1)}}$$

for some constant $A$. It is clear from the symmetry shown in Figure 3 that we will have $f(0) = 0$; moreover, since $f$ maps the segment of the real axis $-1 < z < 1$ to the segment $-a < w < a$, we must have $f'(z) > 0$ on $-1 < z < 1$. With our standard conventions the square root in (9) will be positive for $z > t$, have argument $\pi/2$ for $1 < z < t$, and be negative for $-1 < z < 1$; however, it is here more natural to write this square root as $\sqrt{(t^2 - z^2)(1 - z^2)}$ with the convention that it is positive when $|z| < 1$; thus we will have $A > 0$. Finally, using $f(0) = 0$, (5) (or more properly (8)) becomes

$$f(z) = A \int_0^z \frac{d\zeta}{\sqrt{(t^2 - \zeta^2)(1 - \zeta^2)}}, \quad A > 0. \tag{10}$$

The constants $A > 0$ and $t > 1$ must be chosen so that

$$f(1) = A \int_0^1 \frac{d\zeta}{\sqrt{(t^2 - \zeta^2)(1 - \zeta^2)}} = 1,$$

$$f(t) - f(1) = A \int_1^t \frac{d\zeta}{\sqrt{(t^2 - \zeta^2)(\zeta^2 - 1)}} = b, \tag{11}$$

where in both equations of (11) the square roots are positive in the range of integration.

One cannot obtain the integral in (10) in terms of the most standard functions of analysis, but in fact it is so important that it, or rather a closely related integral, has been given a special name: the \textit{incomplete elliptic integral of the first kind}. You can read more about it on the “Elliptic integral” page of Wikipedia, at


Specifically, the Wikipedia page introduces the incomplete elliptic integral as the function

$$F(z; k) = \int_0^z \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k^2\zeta^2)}}, \tag{12}$$
so that \( f(z) = AtF(z; 1/t) \). The integrals in (11) can be expressed in terms of the complete elliptic integral of the first kind, discussed on the same page. Maple knows a lot about elliptic integrals.

When one tries to use Schwarz-Christoffel transformations with a closed polygon of the sort shown in Figure 1 one usually encounters integrals which cannot be evaluated in closed form, as in the above example. However, the method can also be applied to “degenerate” polygons in which one or more more of the vertices lie at infinity, and in some such cases this difficulty does not arise. We will discuss several examples of this type.

**Example 2:** We would like to map the the upper half plane onto half strip \(-a < x < a, y > 0\). To do so we regard the strip as the \( b \to \infty \) limit of a tall isosceles triangle with vertices \( w_1 = -a, w_2 = a, w_3 = ib \); see Figure 4. Rather than try to first write the formula for the Schwarz-Christoffel transformation for the triangle, then take the limit, we use this idea to write down the transformation directly for the half strip. This has vertices \( w_1 = -a, w_2 = a, w_3 = \infty \) and exterior angles \( \pi/2, \pi/2, \) and \( \pi \), respectively, where to get the angle at \( \infty \) we take the limit of the exterior angle at \( ib \) of the triangle (or use \( \sum_{i=1}^{3} \gamma_i = 1 \)). We take \( z_1 = -1, z_2 = 1, \) and \( z_3 = \infty \); again from considerations of symmetry this will lead to \( f(0) = 0 \) and so from (7),

\[
f(z) = \frac{2a}{\pi} \int_{0}^{z} \frac{d\zeta}{\sqrt{1 - \zeta^2}} = \frac{2a}{\pi} \sin^{-1}(z) \tag{13}\]

Here as in Example 1 we define the square root to be positive for \( |z| < 1 \); we have set the overall constant \( A \) in (7) to \( A = 2a/\pi \) in order to have \( f(1) = a \). The inverse mapping from the half strip to the upper half plane is \( z(w) = \sin(\pi w/2a) \). Of course, this answer is already familiar to us from Exercise 2.3.13, but the Schwarz-Christoffel transformation gives us a systematic way to find it.

**Example 3:** In Example 2 of Section 2.6 our text discusses fluid flow over a semi-circular bump. Using a Schwarz-Christoffel transformation we can similarly discuss flow over a “spike” of the sort shown in Figure 5(a). To do this we want a conformal map of the region \( D \) shown in the figure—the upper half plane minus the spike—to the upper half plane. The idea is to think of the region \( D \) as the interior of a degenerate polygon, and thus construct a mapping from the upper half plane to \( D \) as a Schwarz-Christoffel transformation, then take the inverse of this mapping. Essentially, then, we want to have \( D = \lim_{\epsilon \to 0} D_\epsilon \) for an appropriate polygon whose interior is \( D_\epsilon \). There are many ways to do this, and it does not matter much which we choose; one way is shown in Figure 5(b), where the polygon shown has four vertices approaching \( \infty \) and two approaching 0, so that its interior, the region \( D_\epsilon \), grows to the region \( D \).

In practice, however, it is not necessary to go through this process. One simply looks at \( D \) directly as the interior of a degenerate polygon with four vertices; in order as one goes along the boundary, starting from the negative real axis, these are \( w_1 = 0, w_2 = ib \),
$w_3 = 0$, and $w_4 = \infty$ (all four of the “outer” vertices of the polygon in Figure 5(b) have collapsed into a single vertex at $\infty$). The exterior angles of the polygon are respectively $\pi/2$, $-\pi$ (since we turn clockwise $180^\circ$ at $ib$), $\pi/2$, and $2\pi$. (One can understand this exterior angle of $2\pi$ at $\infty$ in several ways: it is the sum of the four exterior angles, each $\pi/2$, of the “outer” vertices in figure 5(b), and a value of $2\pi$ is needed to keep the sum of all the exterior angles equal to $2\pi$.) Thus $\gamma_1 = 1/2$, $\gamma_2 = -1$, $\gamma_3 = 1/2$, and $\gamma_4 = 2$. Again from symmetry it is natural to take $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, and $z_4 = \infty$, leading to

$$f(z) = A \int^z \frac{\zeta}{\sqrt{\zeta^2 - 1}} \, d\zeta + B = A \sqrt{z^2 - 1} + B.$$ 

The condition that $f(1) = f(-1) = 0$ tells us that that we must take $B = 0$; the condition that $f(0) = ib$ tells us that we may take $A = b$ and use a branch of the square root defined with cuts on $\{z \mid z \geq 1\}$ and $\{z \mid z \leq -1\}$ and with $\arg(1 - z^2) = i\sqrt{1 - z^2}$ on $-1 < z < 1$ (where on the right side of this condition we mean the positive square root):

$$w = f(z) = b \sqrt{z^2 - 1}.$$ 

The inverse mapping $z = g(w)$ is $z = g(w) = \sqrt{(w/b)^2 + 1}$, where the square root is taken so that $z$ is in the upper half plane, that is, so that $\text{Im} \, z > 0$. See Figure 6. One may now study fluid flow past this obstacle as in Example 2 of Section 2.6.
Exercises:

1. The strip in which $0 < \Re z < 1$ can be regarded as a degenerate polygon with two vertices, both at infinity. Use (7) to find a (familiar) mapping of the upper half plane to this strip.

2. Find a mapping of the upper half plane to the triangle $x, y > 0, x + y < 1$ (where $z = x + iy$). Your answer will have to be expressed as an unevaluated integral, but you should calculate the constants $A$ and $B$ in the formula (5) or (7) for the Schwarz-Christoffel transformation in terms of known functions. Hint: if you use (7) rather than (5) (which might be suggested by symmetry) then the constant $A$ will be expressible in terms of the beta function described in Exercise 4.5.14 of Greenberg.

3. Let $D$ be the region obtained by removing from the upper half plane the line $\{x + i | x \leq 0\}$ (see the adjacent figure). Construct a mapping from the upper half plane to $D$; you should be able to find this mapping explicitly. Hint: there are three vertices, two of them at infinity.

4. Let $D'$ be the region obtained by removing from the exterior of the circle $|z| = 1$ the line segment $y = 1$, $0 \leq x \leq 2$. Construct a mapping from the upper half plane to $D'$. Hint: reduce to problem 3 by a bilinear transformation.

5. Let $D$ be the region obtained by removing from the strip $-1 < y < 1$ the origin and the positive real axis. Find a solution $\psi(x, y)$ of Laplace’s equation in $D$ which takes value 0 on the boundaries $y = -1$ and $y = 1$ and value 1 on the positive real axis. In particular, evaluate $\psi(-1, 0)$. (This problem was modified from [1].)

References

Appendix

In this appendix we want to fill in several gaps in our discussion of the Schwarz-Christoffel transformation. Unfortunately, we need to use a little contour integration. If you are not familiar with this tool you should probably just skim the latter part of the appendix until we have covered Chapter 23 of the text.

Recall that we argued that if we want to send the $n$ (finite) points $z_1, \ldots, z_n$ on the real axis to the $n$ vertices $w_1, \ldots, w_n$ of the polygon in the $w$ plane we should use a mapping of the form given in (5), which we repeat here for convenience:

$$f(z) = A \int^z (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_{n-1})^{-\gamma_{n-1}} (\zeta - z_n)^{-\gamma_n} \, d\zeta + B. \quad (A.1)$$

We also asserted, without any argument, that if we wanted to take $z_n = \infty$, that is, to send the (finite) real points $z_1, \ldots, z_{n-1}$ to the vertices $w_1, \ldots, w_{n-1}$ and send $\infty$ to $w_n$, then we should use a mapping of the form (see (7))

$$f(z) = A \int^z (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_{n-1})^{-\gamma_{n-1}} \, d\zeta + B, \quad (A.2)$$

that is, we should just omit from the integrand in (A.1) the factor for $z_n$. We want to justify both of these formulas somewhat more completely.

First, what is the difference between (A.1) and (A.2)? Of course, the integrand in (A.1) involves $n$ factors and that in (A.2) only $n - 1$, but since $n$ is arbitrary this is not a real distinction. Rather, the difference is this: in writing down (A.1) we had a factor $(z - z_i)^{\gamma_i}$ for each vertex $w_i$ of the polygon, which means that the $\gamma_i$ appearing in (A.1) sum to 2, since the exterior angles in any polygon sum to $2\pi$. On the other hand, in (A.2) the integrand has one fewer factor than there are vertices in the polygon to which we are trying to map, which means that the $\gamma_i$ in (A.2) do not sum to 2 (but rather to $2 - \gamma_n$). So for the moment we will consider the two formulas together, that is, we will study the mapping defined by (A.1) but without committing ourselves to the value of $\sum \gamma_i$.

Now, what was our justification of (A.1)? Let us write $w_i = f(z_i)$ (of course in the end we want these $w_i$ to be the vertices of a given polygon, but for the moment we are studying (A.1) by itself). The points $z_1, \ldots, z_n$ divide the real $z$ axis into $n + 1$ segments, and we showed (see (4)) that each of these segments, thought of as oriented from left to right, is mapped to an oriented straight line segment in the $w$ plane whose tangent vector has argument $\theta$ given by

(a) $\theta = \theta_1 = \arg A - \pi \sum_{k=1}^{n} \gamma_k$, for the segment from $-\infty$ to $z_1$;

(b) $\theta = \theta_{i+1} = \arg A - \pi \sum_{k=i+1}^{n} \gamma_k$, for the segment from $z_i$ to $z_{i+1}$, $1 \leq i \leq n - 1$;

(c) $\theta = \theta_{n+1} = \arg A$, for the segment from $z_n$ to $\infty$.

Of course, the segment from $z_i$ to $z_{i+1}$, $1 \leq i \leq n - 1$, considered in (b), must be mapped to the straight line segment in the $w$ plane from $w_i$ to $w_{i+1}$.

We now ask how the first and last segments of the real $z$ axis—that is, the infinite segment from $-\infty$ to $z_1$ and the infinite segment from $z_n$ to $\infty$—are mapped by $f$. We
know of course that these are mapped to straight lines, specifically, the first to a straight line from somewhere to \( w_1 \) in the direction \( \theta_1 \) (see (a)) and the last to a straight line from \( w_n \) to somewhere in the direction \( \theta_{n+1} \) (see (c)). These lines will thus run in the same direction when \( \theta_1 \) and \( \theta_{n+1} \) differ by an integer multiple of \( 2\pi \). Thus we from (a) and (c) we have:

**Fact 1:** The images of the first and last segments run in the same direction if and only if \( \sum_{k=1}^{n} \gamma_i \) is an even integer.

Next we must determine where the images of these two segments come from (for the first segment) or go to (for the last), that is, we want to find

\[
w_0 = \lim_{a \to -\infty} f(a) \quad \text{and} \quad w_{n+1} = \lim_{b \to \infty} f(b)
\]  

(A.3)

These limits may or may not exist as finite complex numbers; if not, we can reasonably assign them the value \( \infty \). Our goal is to rule out situations like those in Figure A.1 (drawn for \( n = 4 \)). Notice that in Figure A.1(a), \( w_5 = \infty \) but \( w_0 \) is finite; in Figure A.1(b) \( w_0 \) and \( w_5 \) are both finite but not equal, and in Figure A.1(c) the initial and final segments of the image are in the same direction, because \( \sum_{i=1}^{4} \gamma_i = 2 \), but the “polygon” still does not close.

Figure A.1(a)  
Figure A.1(b)  
Figure A.1(c)

Consider first \( w_{n+1} \). If we choose some point \( z_+ \) on the real axis satisfying \( z_+ > z_{n+1} \) then from (A.1) and (A.3),

\[
w_{n+1} = f(z_+) + \lim_{b \to \infty} b \left( f(b) - f(z_+) \right) 
\]

\[
= f(z_+) + \lim_{b \to \infty} b \left( z - z_1 \right)^{-\gamma_1} \left( z - z_2 \right)^{-\gamma_2} \cdots \left( z - z_n \right)^{-\gamma_n} d\zeta.
\]  

(A.4)

Now \( w_{n+1} \) will be finite if the integral in (A.4) converges and equal to \( \infty \) if it diverges. For very large \( \zeta \) the integrand is approximately \( \zeta^{-\sum_{i=1}^{n} \gamma_i} \), and since \( \int_{-\infty}^{\infty} \zeta^{-\alpha} d\zeta \) converges if \( \alpha > 1 \) and diverges otherwise, the integral in (A.4) converges if and only if \( \sum_{i=1}^{n} \gamma_i > 1 \). A similar analysis holds for \( w_0 \), and we have:

**Fact 2:** \( w_0 \) and \( w_{n+1} \) are both finite if \( \sum_{i=1}^{n} \gamma_i > 1 \) and both equal to \( \infty \) otherwise.

This rules out the situation shown in Figure A.1(a).
Suppose finally that $w_0$ and $w_{n+1}$ are finite, that is, that $\sum_{i=1}^{n} \gamma_i > 1$. We want to show that $w_0 = w_{n+1}$. If we let $g(\zeta) = \prod_{i=1}^{n} (\zeta - z_i)^{-\gamma_i}$ denote the integrand in (A.1) we have

$$w_{n+1} - w_0 = \lim_{a \to \infty} \int_{C_a} g(\zeta) \, d\zeta,$$

where $C_a$ is the contour shown in Figure A.2(a). Now by Cauchy’s Theorem,

$$\int_{C_a} g(\zeta) \, d\zeta + \int_{C_a'} g(\zeta) \, d\zeta = 0,$$

where $C_a'$ is the half-circle shown in Figure A.2(b). We want to obtain an $ML$ bound for $\int_{C_a} g(\zeta) \, d\zeta$; $L$ is $\pi a$ and if $a \geq 2 \max\{|z_1|, \ldots, |z_n|\}$ then for $\zeta$ on $C_a$,

$$|\zeta - z_i| \geq a - |z_i| \geq a - \frac{a}{2} = \frac{a}{2}, \quad \text{and so} \quad |g(\zeta)| \leq \left(\frac{2}{a}\right)^{\Sigma \gamma_i}.$$

Thus we have from (A.6),

$$\left|\int_{C_a} g(\zeta) \, d\zeta\right| = \left|\int_{C_a'} g(\zeta) \, d\zeta\right| \leq \left(\frac{2}{a}\right)^{\Sigma \gamma_i} \pi a = \pi 2^{\Sigma \gamma_i} a^{1-\Sigma \gamma_i},$$

and (A.7) approaches 0 as $a \to \infty$ because $\sum \gamma_i > 1$. By (A.6), then, $\lim_{a \to \infty} \int_{C_a} g(\zeta) \, d\zeta = 0$, and thus from (A.5) we have that $w_0 = w_{n+1}$. We have established

**Fact 3:** If $w_0$ and $w_{n+1}$ are finite then they are equal.

This rules out the situations shown in Figure A.1(b) and Figure A.1(c).

We can now summarize the consequences of our reasoning. Consider formula (A.1) in two cases.

**Case 1:** $\sum_{i=1}^{n} \gamma_i = 2$ : This is the situation we originally considered; see (5). We now know from Facts 2 and 3 that $w_0$ and $w_{n+1}$ are finite and equal, and from Fact 1 that the segments from $w_0 (= w_{n+1})$ to $w_1$ and from $w_n$ to $w_{n+1} (= w_0)$ have the same direction. Thus the situation must be as in Figure A.3(a); the point $w_0 = w_{n+1}$ is a point on one side of the polygon, not a true vertex.

**Case 2:** $\sum_{i=1}^{n} \gamma_i \neq 2$ : This is the situation in which we want $z_{n+1} = \infty$ to become one of the vertices in the $w$ plane; see (7). We now know from Facts 2 and 3 that $w_0$ and $w_{n+1}$
are either both infinite or both finite, and that in the latter case they are equal. Consider then two subcases:

- **(i)** If \( w_0 = w_{n+1} \) is finite then \( \sum_{i=1}^{n} \gamma_i > 1 \) by Fact 2. This sum is not a multiple of 2, because by our assumption in Case 2 it is not equal to 2 (and I cannot see how it could be 4, 6, ...), so that from Fact 1 the segments from \( w_0(= w_{n+1}) \) to \( w_1 \) and from \( w_n \) to \( w_{n+1}(= w_0) \) have different directions. Thus the situation must be as in Figure A.3(b); the point \( w_0 = w_{n+1} \) is a (finite) vertex of the image polygon.

- **(ii)** If \( w_0 = w_{n+1} = \infty \) there is not too much to say; the “extra” vertex into which \( z = \infty \) maps is at \( \infty \). The directions of the sides of the polygon running to and from this infinite vertex will be in the same direction if \( \sum \gamma_i = 0 \) and different otherwise. Example 2 above (See Figure 4) is an example in which these directions are different (they differ by \( \pi \)); Example 3 (See Figure 5(a)) is one in which they are the same.

![Figure A.3(a)](image1.png)  ![Figure A.3(b)](image2.png)

To completely justify (5) and (7) we should show that by proper choice of the points \( z_1, \ldots, z_n \) we can obtain as image any polygon in the \( w \) plane, that is, any choice of \( w_1, \ldots, w_n \). I have never seen a proof of this and will not try to construct one here.