Appendix: More on Schwarz-Christoffel transformations

In this appendix we want to fill in several gaps in our discussion of the SchwarzChristoffel transformation.

Recall that we argued that if we want to send the $n$ (finite) points $z_1, \ldots, z_n$ on the real axis to the $n$ vertices $w_1, \ldots, w_n$ of the polygon in the $w$ plane we should use a mapping of the form

$$f(z) = A \int^z (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_{n-1})^{-\gamma_{n-1}} (\zeta - z_n)^{-\gamma_n} d\zeta + B.$$  \hfill (A.1)

We also asserted, without any argument, that if we wanted to take $z_n = \infty$, that is, to send the (finite) real points $z_1, \ldots, z_{n-1}$ to the vertices $w_1, \ldots, w_{n-1}$ and send $\infty$ to $w_n$, then we should use a mapping of the form

$$f(z) = A \int^z (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_{n-1})^{-\gamma_{n-1}} d\zeta + B,$$ \hfill (A.2)

that is, we should just omit from the integrand in (A.1) the factor for $z_n$. We want to justify both of these formulas somewhat more completely.

First, what is the difference between (A.1) and (A.2)? Of course, the integrand in (A.1) involves $n$ factors and that in (A.2) only $n - 1$, but since $n$ is arbitrary this is not a real distinction. Rather, the difference is this: in writing down (A.1) we had a factor $(z - z_i)^{\gamma_i}$ for each vertex $w_i$ of the polygon, which means that the $\gamma_i$ appearing in (A.1) sum to $2$, since the exterior angles in any polygon sum to $2\pi$. On the other hand, in (A.2) the integrand has one fewer factor than there are vertices in the polygon to which we are trying to map, which means that the $\gamma_i$ in (A.2) do not sum to 2 (but rather to $2 - \gamma_n$).

So for the moment we will consider the two formulas together, that is, we will study the mapping defined by (A.1) but without committing ourselves to the value of $\sum \gamma_i$.

Now, what was our justification of (A.1)? Let us write $w_i = f(z_i)$ (of course in the end we want these $w_i$ to be the vertices of a given polygon, but for the moment we are studying (A.1) by itself). The points $z_1, \ldots, z_n$ divide the real $z$ axis into $n + 1$ segments, and we showed (see the formula just above (4)) that each of these segments, thought of as oriented from left to right, is mapped to an oriented straight line segment in the $w$ plane whose tangent vector has argument $\theta$ given by

(a) $\theta = \theta_1 = \arg A - \pi \sum_{k=1}^n \gamma_k$, for the segment from $-\infty$ to $z_1$;
(b) $\theta = \theta_{i+1} = \arg A - \pi \sum_{k=i+1}^n \gamma_k$, for the segment from $z_i$ to $z_{i+1}$, $1 \leq i \leq n - 1$;
(c) $\theta = \theta_{n+1} = \arg A$, for the segment from $z_n$ to $\infty$.

Of course, the segment from $z_i$ to $z_{i+1}$, $1 \leq i \leq n - 1$, considered in (b), must be mapped to the straight line segment in the $w$ plane from $w_i$ to $w_{i+1}$.

We now ask how the first and last segments of the real $z$ axis—that is, the infinite segment from $-\infty$ to $z_1$ and the infinite segment from $z_n$ to $\infty$—are mapped by $f$. We know of course that these are mapped to straight lines, specifically, the first to a straight line from somewhere to $w_1$ in the direction $\theta_1$ (see (a)) and the last to a straight line from
$w_n$ to somewhere in the direction $\theta_{n+1}$ (see (c)). These lines will thus run in the in the same direction when $\theta_1$ and $\theta_{n+1}$ differ by an integer multiple of $2\pi$. Thus we from (a) and (c) we have:

**Fact 1:** The images of the first and last segments run in the in the same direction if and only if $\sum_{k=1}^{n} \gamma_i$ is an even integer.

Next we must determine where the images of these two segments come from (for the first segment) or go to (for the last), that is, we want to find

$$w_0 = \lim_{a \to -\infty} f(a) \quad \text{and} \quad w_{n+1} = \lim_{b \to \infty} f(b) \quad \text{(A.3)}$$

These limits may or may not exist as finite complex numbers; if not, we can reasonably assign them the value $\infty$. Our goal is to rule out situations like those in Figure A.1 (drawn for $n = 4$). Notice that in Figure A.1(a), $w_5 = \infty$ but $w_0$ is finite; in Figure A.1(b) $w_0$ and $w_5$ are both finite but not equal, and in Figure A.1(c) the initial and final segments of the image are in the same direction, because $\sum_{i=1}^{4} \gamma_i = 2$, but the “polygon” still does not close.

Consider first $w_{n+1}$. If we choose some point $z_+$ on the real axis satisfying $z_+ > z_{n+1}$ then from (A.1) and (A.3),

$$w_{n+1} = f(z_+) + \lim_{b \to \infty} \left( f(b) - f(z_+) \right)$$

$$= f(z_+) + \lim_{b \to \infty} A \int_{z_+}^{b} (\zeta - z_1)^{-\gamma_1} (\zeta - z_2)^{-\gamma_2} \cdots (\zeta - z_n)^{-\gamma_n} d\zeta. \quad \text{(A.4)}$$

Now $w_{n+1}$ will be finite if the integral in (A.4) converges and equal to $\infty$ if it diverges.

For very large $\zeta$ the integrand is approximately $\zeta^{-\sum_{i=1}^{n} \gamma_i}$, and since $\int_{-\infty}^{\infty} \zeta^{-\alpha} d\zeta$ converges if $\alpha > 1$ and diverges otherwise, the integral in (A.4) converges if and only if $\sum_{i=1}^{n} \gamma_i > 1$. A similar analysis holds for $w_0$, and we have:

**Fact 2:** $w_0$ and $w_{n+1}$ are both finite if $\sum_{i=1}^{n} \gamma_i > 1$ and both equal to $\infty$ otherwise

This rules out the situation shown in Figure A.1(a).

Suppose finally that $w_0$ and $w_{n+1}$ are finite, that is, that $\sum_{i=1}^{n} \gamma_i > 1$. We want to show that $w_0 = w_{n+1}$. If we let $g(\zeta) = \prod_{i=1}^{n} (\zeta - z_i)^{-\gamma_i}$ denote the integrand in (A.1) we have

$$w_{n+1} - w_0 = \lim_{a \to -\infty} \int_{C_a} g(\zeta) d\zeta, \quad \text{(A.5)}$$

A.2
where $C_α$ is the contour shown in Figure A.2(a). Now by Cauchy’s Theorem,

\[
\int_{C_α} g(ζ) dζ + \int_{C_α'} g(ζ) dζ = 0, \quad (A.6)
\]

where $C_α'$ is the half-circle shown in Figure A.2(b). We want to obtain an $ML$ bound for $\int_{C_α} g(ζ) dζ$; $L$ is $πa$ and if $a ≥ 2 \max\{|z_1|, \ldots, |z_n|\}$ then for $ζ$ on $C_α'$,

\[
|ζ − z_i| ≥ a − |z_i| ≥ a − \frac{a}{2} = \frac{a}{2}, \quad \text{and so} \quad |g(ζ)| ≤ \left(\frac{2}{a}\right) \sum_1^n γ_i.
\]

Thus we have from (A.6),

\[
\left|\int_{C_α} g(ζ) dζ\right| = \left|\int_{C_α'} g(ζ) dζ\right| ≤ \left(\frac{2}{a}\right)^{n+1} \pi a = \pi 2^{n+1} a^{1−n}, \quad (A.7)
\]

and (A.7) approaches 0 as $a → ∞$ because $\sum_1^n γ_i > 1$. By (A.6), then, $\lim_{a→∞} \int_{C_α} g(ζ) dζ = 0$, and thus from (A.5) we have that $w_0 = w_{n+1}$. We have established

**Fact 3:** If $w_0$ and $w_{n+1}$ are finite then they are equal.

This rules out the situations shown in Figure A.1(b) and Figure A.1(c).

We can now summarize the consequences of our reasoning. Consider formula (A.1) in two cases.

**Case 1:** $\sum_1^n γ_i = 2$ : This is the situation we originally considered; see (4). We now know from Facts 2 and 3 that $w_0$ and $w_{n+1}$ are finite and equal, and from Fact 1 that the segments from $w_0(= w_{n+1})$ to $w_1$ and from $w_n$ to $w_{n+1}(= w_0)$ have the same direction. Thus the situation must be as in Figure A.3(a); the point $w_0 = w_{n+1}$ is a point on one side of the polygon, not a true vertex.

**Case 2:** $\sum_1^n γ_i ≠ 2$ : This is the situation in which we want $z_{n+1} = ∞$ to become one of the vertices in the $w$ plane; see (6). We now know from Facts 2 and 3 that $w_0$ and $w_{n+1}$ are either both infinite or both finite, and that in the latter case they are equal. Consider then two subcases:

- **(i)** If $w_0 = w_{n+1}$ is finite then $\sum_1^n γ_i > 1$ by Fact 2. This sum is not a multiple of 2, because by our assumption in Case 2 it is not is not equal to 2 (and I cannot see
how it could be 4, 6, . . . , so that from Fact 1 the segments from $w_0(= w_{n+1})$ to $w_1$ and from $w_n$ to $w_{n+1}(= w_0)$ have different directions. Thus the situation must be as in Figure A.3(b); the point $w_0 = w_{n+1}$ is a (finite) vertex of the image polygon.

- (ii) If $w_0 = w_{n+1} = \infty$ there is not too much to say; the “extra” vertex into which $z = \infty$ maps is at $\infty$. The directions of the sides of the polygon running to and from this infinite vertex will be in the same direction if $\sum \gamma_i = 0$ and different otherwise. Example 2 above (See Figure 4) is an example in which these directions are different (they differ by $\pi$); Example 3 (See Figure 5(a)) is one in which they are the same.

To completely justify (4) and (6) we should show that by proper choice of the points $z_1, \ldots, z_n$ we can obtain as image any polygon in the $w$ plane, that is, any choice of $w_1, \ldots, w_n$. I have never seen a proof of this and will not try to construct one here.