Note: The is the first exam from Fall 2010, with one extra problem.

1. A function $f(t)$ is defined for $t \geq 0$ by $f(t) = \begin{cases} 0, & \text{if } t < 1, \\ (t-1)^2, & \text{if } 1 \leq t < 3, \\ 4, & \text{if } t \geq 3. \end{cases}$

Express $f(t)$ in terms of a single formula using the Heaviside function, then find its Laplace transform.

2. Suppose that $f(t) = \displaystyle\int_0^t e^u \sin 3u \cosh 3(t-u)\, du$. Find $L\{f(t)\}$.

3. (a) Use the Laplace transform to solve the initial value problem

$$x'' + 3x' - 4x = A\delta(t-2), \quad x(0) = 0, \quad x'(0) = 2,$$

where A is a constant.

(b) Determine a value for A, if one exists, such that $\lim_{t\to\infty} x(t) = 0$.

4. Find the power series expansion (Taylor series) of $f(x) = \dfrac{1}{3+2x}$ with center $x_0 = 1$, and give its radius of convergence.

5. (a) Suppose that $\nu > 0$ and that ν is not an integer. Evaluate $\lim_{x\to 0} x^\nu J_{-\nu}(x)$. Your answer will involve the gamma function. (There is a formula for $J_{-\nu}$ on the formula sheet.)

(b) Suppose that $\nu = 3/2$. Simplify your answer in (a) to a form not involving the gamma function.

6. Consider the differential equation

$$(1-x)y'' + 2xy' + 3y = 0.$$

(a) Show that $x = 1$ is a regular singular point of this equation.

(b) Find the corresponding indicial equation. Hint: $r(r-1) + p_0 r + q_0 = 0$.

(c) Give the *form* of the two Frobenius solutions associated with this singular point.

(d) For what values of x will these solutions necessarily be defined? Justify your answer.

7. (a) Consider the differential equation

$$x^2y'' + x^2y' - 2(1+x)y = 0.$$

It is a fact, which you do not have to verify, that $x = 0$ is a regular singular point of this equation and that the corresponding indicial equation is $r^2 - r - 2 = (r+1)(r-2) = 0$.

(a) Verify that $y(x) = x^2$ is one solution of this equation. (Hint: this is *very* easy.)

A second solution will have the form $y_2(x) = Cx^2 \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n$, with $b_0 = 1$.

(b) One of the coefficients b_n may be assumed to be zero. Which one? ____

(c) Find C and b_n, $n = 2, \ldots, 4$. Use the assumption mentioned in (b).

(d) Find a general formula for b_n, $n > 4$.

8. Express the general solution of the equation

$$x^2y'' + xy' + xy = 0$$

in terms of Bessel functions. (Since this is a second order equation, the general solution will contain two undetermined constants.)

(b) Find the values of the constants in your solution above which yield a solution $y(x)$ with the property that $\lim_{x\to 0} y(x) = 3$.