

Taylor series (with radii of convergence given):

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_0^{\infty} x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_0^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_0^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

**The Gamma function.** For  $x > 0$ ,  $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ .

If  $x$  is not 0 or a negative integer,  $\Gamma(x+1) = x\Gamma(x)$ .

If  $n$  is a non-negative integer,  $\Gamma(n+1) = n!$ .  $\Gamma(1/2) = \sqrt{\pi}$ .

**The Method of Frobenius—solution forms:**

$$y_1(x) : x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) : x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad \text{or} \quad y_1(x)(\ln x) + x^{r_2} \sum_{n=1}^{\infty} c_n x^n \quad \text{or} \quad C y_1(x)(\ln x) + x^{r_2} \sum_{n=0}^{\infty} c_n x^n.$$

**The Method of Frobenius—useful formula:**

$$u'' + p(x)u' + q(x)u = \frac{C}{x^2} [y_1(x) - xp(x)y_1(x) - 2xy_1'(x)].$$

**Bessel Functions.**

**A. The Bessel equation of order  $\nu$ :**  $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ .

**B. If  $u$  solves the Bessel equation of order  $\nu$ , and  $b > 0$ , then**

$$y(x) = x^{\nu/\alpha} u(\alpha\sqrt{bx}^{1/\alpha}) \quad \text{solves} \quad y'' + \frac{a}{x}y' + bx^{c-a}y = 0,$$

where

$$\alpha = \frac{2}{c-a+2} \quad \text{and} \quad \nu = \frac{1-a}{c-a+2}.$$

**C. Bessel functions:**

$$J_{\nu}(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k}$$

$$Y_{\nu}(x) = \frac{(\cos \nu\pi)J_{\nu}(x) - J_{-\nu}(x)}{\sin \nu\pi}, \quad \text{if } \nu \neq 0, 1, 2, \dots$$

$$Y_n(x) = \lim_{\nu \rightarrow n} Y_{\nu}(x), \quad \text{if } n = 0, 1, 2, \dots$$

**The Fourier transform:**

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx, \quad \mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x} d\omega.$$

**d'Alembert's solution:**

$$u(x, t) = F(x-ct) + G(x+ct) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) dz$$

Various Fourier-type expansions:

We write  $f(x) \sim$  series to indicate that the given series is some Fourier-type expansion of  $f(x)$ . All the formulas for coefficients come from the formula

$$c_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$$

with  $\varphi_1, \varphi_2, \dots$  a complete orthogonal set for  $C_p[-\ell, \ell]$  or  $C_p[0, L]$ .

$$\left. \begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right]; \\ a_0 &= \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) dx, \quad a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx. \end{aligned} \right\} \quad (1)$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}; \quad c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell} dx. \quad (2)$$

$$\left. \begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad (0 < x < L); \\ a_0 &= \frac{1}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \end{aligned} \right\} \quad (3)$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (0 < x < L); \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \quad (4)$$

$$f(x) \sim \sum_{n=1,3,5,\dots} a_n \cos \frac{n\pi x}{2L} \quad (0 < x < L); \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{2L} dx. \quad (5)$$

$$f(x) \sim \sum_{n=1,3,5,\dots} b_n \sin \frac{n\pi x}{2L} \quad (0 < x < L); \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} dx. \quad (6)$$

Some antiderivatives:

$$\begin{aligned} \int e^{-ax} \sin(bx) dx &= -\frac{e^{-ax}}{a^2 + b^2} (a \sin(bx) + b \cos(bx)) + C \\ \int e^{-ax} \cos(bx) dx &= \frac{e^{-ax}}{a^2 + b^2} (-a \cos(bx) + b \sin(bx)) + C \\ \int x \cos(bx) dx &= \frac{\cos(bx)}{b^2} + \frac{x \sin(bx)}{b} + C & \int x \sin(bx) dx &= \frac{\sin(bx)}{b^2} - \frac{x \cos(bx)}{b} + C \end{aligned}$$

Some trig identities:

$$\begin{aligned} \cos^2 x &= \frac{1 + \cos(2x)}{2} & \sin^2 x &= \frac{1 - \cos(2x)}{2} \\ \cos(A) \cos(B) &= (1/2) [\cos(A+B) + \cos(A-B)] \\ \sin(A) \cos(B) &= (1/2) [\sin(A+B) + \sin(A-B)] \\ \sin(A) \sin(B) &= (1/2) [\cos(A-B) - \cos(A+B)] \end{aligned}$$

Sturm-Liouville problem:

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \quad \langle f(x), y(x) \rangle_w = \int_0^L f(x)g(x)w(x) dx$$