Various Fourier-type expansions:

We write $f(x) \sim$ series to indicate that the given series is some Fourier-type expansion of $f(x)$. All the formulas for coefficients come from the formula

$$c_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$$

with $\varphi_1, \varphi_2, \ldots$ a complete orthogonal set for $C_p[-\ell, \ell]$ or $C_p[0, L]$.

$$\left.\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right]; \\ a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx, \quad a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx. \end{aligned}\right\} \qquad \textbf{(1)}$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}; \qquad c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\,dx. \qquad \textbf{(2)}$$

$$\left.\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad (0 < x < L); \\ a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx. \end{aligned}\right\} \qquad \textbf{(3)}$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad (0 < x < L); \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \qquad \textbf{(4)}$$

$$f(x) \sim \sum_{n=1,3,5,\ldots} a_n \cos\frac{n\pi x}{2L} \quad (0 < x < L); \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx. \qquad \textbf{(5)}$$

$$f(x) \sim \sum_{n=1,3,5,\ldots} b_n \sin\frac{n\pi x}{2L} \quad (0 < x < L); \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx. \qquad \textbf{(6)}$$

Some antiderivatives:

$$\int e^{-ax}\sin(bx)\,dx = -\frac{e^{-ax}}{a^2+b^2}\left(a\sin(bx) + b\cos(bx)\right) + C$$

$$\int e^{-ax}\cos(bx)\,dx = \frac{e^{-ax}}{a^2+b^2}\left(-a\cos(bx) + b\sin(bx)\right) + C$$

$$\int x\cos(bx)\,dx = \frac{\cos(bx)}{b^2} + \frac{x\sin(bx)}{b} + C \qquad \int x\sin(bx)\,dx = \frac{\sin(bx)}{b^2} - \frac{x\cos(bx)}{b} + C$$

Some trig identities:

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos(A)\cos(B) = (1/2)\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin(A)\cos(B) = (1/2)\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin(A)\sin(B) = (1/2)\left[\cos(A-B) - \cos(A+B)\right]$$