2.5 Example: The heat equation in a disk

In this section we study the two-dimensional heat equation in a disk, since applying separation of variables to this problem gives rise to both a periodic and a singular Sturm-Liouville problem. The equation is

$$\alpha^2 \nabla^2 u(x,y,t) = \frac{\partial}{\partial t} u(x,y,t), \qquad x^2+y^2 \le a^2;$$

here a is the radius of the disk (whose center is assumed to be located at the origin of our coordinate system) and b∇^2, also written as $\triangle$, is the two-dimensional Laplacian, that is, $\nabla^2 u = u_{xx} + u_{yy}$. We will impose a homogeneous Dirichlet boundary condition at the boundary of the disk, i.e., require that $u(x,y,t) = 0$ if $x^2 + y^2 = a^2$; this could easily be replaced by a Neumann or Robin boundary condition. We will also impose an initial condition specifying the temperature distribution at time $t = 0$.

Since the domain and the PDE have rotational symmetry it is natural to work in polar coordinates (r,θ), where $x = r\cos\theta$ and $y = r\sin\theta$. We must then rewrite the x and y derivatives contained in ∇^2 as derivatives with respect to r and θ. The result is derived in Section 16.7 of our text by Greenberg, with the final result appearing in equation (18), which to avoid confusion with equations in these notes we will write as (G.16.7.18). That equation refers to cylindrical coordinates, but to pass to polar coordinates one simply neglects all derivatives with respect to z. The r derivatives appearing in (G.16.7.18) can conveniently be rewritten as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Thus our problem becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}, \qquad 0 \le r < a, \quad 0 \le \theta < 2\pi, \quad t > 0; \tag{2.56}$$

$$u(a,\theta,t) = 0, \qquad 0 \le \theta < 2\pi, t > 0; \tag{2.57}$$

$$u(r,\theta,0) = f(r,\theta), \qquad 0 \le r < a, \quad 0 \le \theta < 2\pi. \tag{2.58}$$

To attack this problem we will first find, by separation of variables, solutions of the PDE (2.56) that also satisfy the boundary conditions (2.57); then we will form a solution that also satisfies the initial condition (2.58) as a superposition of these product solutions.

To apply the separation of variables method we look for solutions of (2.56) in the form

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t).$$

Inserting this proposed solution into (2.56) leads to

$$\frac{\Theta T}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{RT}{r^2}\frac{d^2\Theta}{d\theta^2} = \frac{R\Theta}{\alpha^2}\frac{dT}{dt}.$$

We divide this equation by $R\Theta T$ to obtain

$$\frac{1}{r}\frac{(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{\alpha^2}\frac{T'}{T}. \tag{2.59}$$

The left hand side of (2.59) depends only on r and θ, while the right hand side depends only on t; this is possible only if both sides are constant. Let us call this constant $-\lambda$; then we have

$$\frac{1}{r}\frac{(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda \quad \text{and} \quad \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda. \tag{2.60}$$

The second equation in (2.60) is easily solved for $T(t)$:

$$T(t) = e^{-\lambda\alpha^2 t}. \tag{2.61}$$

Remark 2.7: Let us discuss more carefully why (2.60) follows from (2.59). Equation (2.59) has the form $G(r,\theta) = H(t)$, and this equality is supposed to hold for all r, θ, and t. Now choose any fixed values r_1 and θ_1 of the variables r and θ, and define λ by $G(r_1,\theta_1) = -\lambda$. But now for *any* t, $H(t) = G(r_1,\theta_1) = -\lambda$, which is just the second equation in (2.60). The first equation in (2.60) follows similarly.

Now we turn to the first equation in (2.60) and again separate variables—that is, we write this equation in the form

$$-\frac{\Theta''}{\Theta} = \lambda r^2 + r\frac{(rR')'}{R}. \tag{2.62}$$

The left hand side of (2.62) depends only on θ and the right hand side only on r, so that each must be constant:

$$-\frac{\Theta''}{\Theta} = \mu \quad \text{and} \quad \lambda r^2 + r\frac{(rR')'}{R} = \mu. \tag{2.63}$$

Thus we now have two more ODE's to solve. As we will see, each of these is a Sturm-Liouville problem of a special type.

To solve the first equation in (2.63) for $\Theta(\theta)$, and to determine μ in the process, we need to impose suitable boundary conditions. Now θ, the polar angle, can be thought of as satisfying $0 \le \theta < 2\pi$, but the endpoints of this interval are not really boundaries; nothing special happens at these values of θ. In fact, we should identify all values of θ differing by multiples of 2π, since all the points $\theta + 2n\pi$, $n = 0, \pm1, \pm2, \ldots$, represent the same physical ray in the plane. Because of this, we would like $\Theta(\theta)$ to be a periodic function of θ with period 2π, so that $\Theta(\theta + 2n\pi) = \Theta(\theta)$ for all n. We can achieve this by imposing *periodic boundary conditions*, leading to the problem

$$\Theta'' + \mu\Theta = 0, \quad 0 < \theta < 2\pi, \tag{2.64a}$$

$$\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi) \tag{2.64b}$$

We will see shortly that these boundary conditions will force the solutions of (2.64) to be periodic functions with period 2π.

Remark 2.8: Problem (2.64) is a special case of a *Sturm-Liouville* problem. Recall the *regular* Sturm-Liouville problem studied in Section 17.7:

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \quad a < x < b, \tag{2.65a;}$$

$$\alpha y(a) + \beta y(a) = 0, \qquad \gamma y(b) + \delta y(b) = 0; \tag{2.65b}$$

here

(i) $\infty < a < b < \infty$,

(ii) $w(x)$, $p(x)$, $p'(x)$, and $q(x)$ are real-valued and continuous on $[a, b]$,

(iii) $p(x)$ and $w(x)$ are strictly positive on $[a, b]$, and

(iv) α, β, γ, and δ are real, and neither both α and β, nor both γ and δ, are zero.

The boundary conditions (2.65b) are called *separated*, since there is one boundary condition at a and an independent one at b. Now our PDE (2.64a) is a special case of (2.65a), obtained by taking $a = 0$, $b = 2\pi$, $p(x) = w(x) = 1$, and $q(x) = 0$, but the boundary conditions link the values of y and y' at a and b. These are called *periodic* boundary conditions, and the problem (2.64) is a *periodic Sturm-Liouville problem*.

To solve (2.64) we consider separately the cases $\mu = 0$, $\mu > 0$, and $\mu < 0$;

Case A: $\mu = 0$**.** The general solution of $\Theta'' = 0$ is $\Theta(\theta) = A + B\theta$, and the boundary condition $\Theta(0) = \Theta(2\pi)$ requires that $B = 0$. The remaining solution $\Theta = A$ satisfies both boundary conditions. The factor A is irrelevant to our purposes, since we are looking only for one solution; later we will restore this constant when we form a linear combination of the solutions that we have found. So we take

$$\mu_0 = 0, \qquad \Theta_0(\theta) = 1. \tag{2.66}$$

Case B: $\mu > 0$**.** The general solution of (2.64a) is now $\Theta(\theta) = A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$, and the boundary conditions (2.64b) imply that

$$A = A\cos 2\pi\sqrt{\mu} + B\sin 2\pi\sqrt{\mu}, \qquad B = -A\sin\pi\sqrt{\mu} + B\cos 2\pi\sqrt{\mu},$$

or

$$\begin{bmatrix} \cos 2\pi\sqrt{\mu} - 1 & \sin 2\pi\sqrt{\mu} \\ -\sin\pi\sqrt{\mu} & \cos 2\pi\sqrt{\mu} - 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \mathbf{0}.$$

This homogeneous linear system can have a nonzero solution only if the determinant of the coefficient matrix vanishes, which leads to $\cos 2\pi\sqrt{\mu} = 1$, so that $\sqrt{\mu} = n$ for some integer n. The case $n = 0$, where $\mu = 0$, was discussed separately in case A. The boundary conditions (2.64b) are automatically satisfied, and we find

$$\mu_n = n^2, \quad \Theta_{n,1} = \cos n\theta, \quad \Theta_{n,2} = \sin n\theta, \quad n = 1, 2, \ldots. \tag{2.67}$$

Case C: $\mu < 0$**.** The solution of (2.64b) is now $\Theta(\theta) = A\cosh\sqrt{-\mu}\,\theta + B\sinh\sqrt{-\mu}\theta$. The analysis proceeds as in Case B, but now we find that μ must satisfy $\cosh 2\pi\sqrt{-\mu} = 1$; the only solution of this equation is $\mu = 0$, which was considered separately in Case A, so there are no negative eigenvalues.

Note that, as promised, the eigenfunctions given in (2.66) and (2.67) are all periodic with period 2π.

Remark 2.9: For the regular Sturm-Liouville problem discussed in Section 17.7, all eigenvalues are simple, that is, for each eigenvalue there is precisely one eigenfunction (up to multiplication by a constant). We see from (2.67) that in a periodic Sturm-Liouville problem the eigenvalues can be degenerate.

Now that we have the eigenvalues (2.66) and (2.67) for the Θ problem (2.64) we return to the problem of finding λ and the corresponding functions $R(r)$. The differential equation to be solved is the second equation in (2.63), which is to hold for $0 < r < a$. The boundary condition (2.57) that $u(r,\theta,t)$ vanish on the boundary $r = 1$ of the disk obviously implies that we should require that $R(a) = 0$. But what boundary condition should we impose at $r = 0$? This corresponds to the center of the disk and is not really a boundary in the original problem; all we want to require is that our solution be well-behaved—say, bounded—at the center. Does this make sense as a boundary condition for $R(r)$?

Remark 2.10: To understand the situation let us write our DE (2.63) in a form which looks like the standard Sturm-Liouville form (2.65a). If we are studying the eigenvalue $\mu_n = n^2$ from the periodic Sturm-Liouville problem for Θ then this form is :

$$[rR']' - \frac{n^2}{r}R + \lambda r R = 0. \tag{2.68}$$

We see that this is of the same form as (2.65a), with $p(r) = r$, $q(r) = -n^2/r$, and $w(r) = r$. But notice that $p(0) = 0$, whereas in the regular Sturm-Liouville problem we required p to be strictly positive everywhere (see Remark 2.8, condition (iii)). We say the the Sturm-Liouville problem with equation (2.68) is *singular* at $r = 0$. For such a problem we cannot usually require a boundary condition of the type (2.65b); rather, requiring that $R(r)$ be bounded near $r = 0$ is a natural condition. This is because $r = 0$ is a singular point of the differential equation (2.68) in the sense of Chapter 4, and one of two linearly independent solutions of such an equation will usually be unbounded at $r = 0$.

Thus, multiplying (2.68) by r and expanding the derivatives, we find that for each value of n, $n = 0, 1, 2, \ldots$, we must solve the singular Sturm-Liouville problem

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0, \quad 0 < r < a; \tag{2.69a}$$

$$R(a) = 0, \qquad R(r) \text{ bounded as } r \to 0. \tag{2.69b}$$

In fact all solutions of this problem have $\lambda > 0$; we will assume that this is true without verifying it. Then the change of variables $s = r\sqrt{\lambda}$ transforms (2.69a) to the Bessel equation of order n:

$$s^2\ddot{R} + s\dot{R} + (s^2 - n^2)R = 0$$

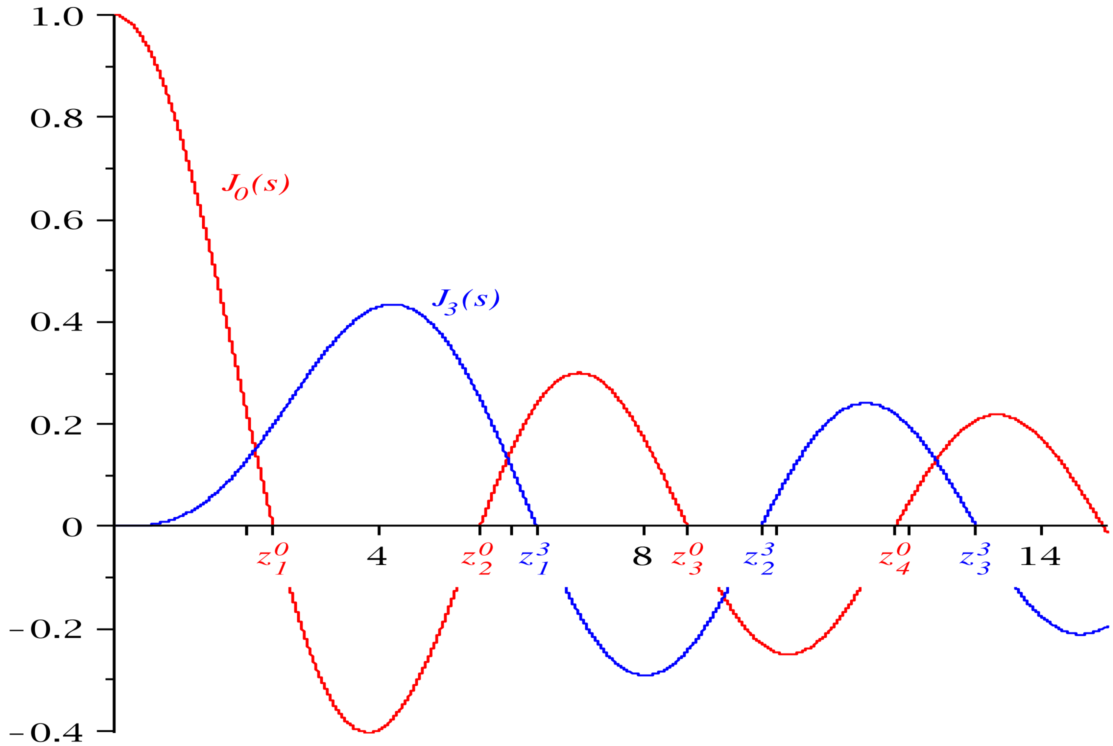

Figure 2.3: The Bessel functions $J_0(s)$ and $J_3(s)$ for $s \geq 0$. The first four zeros of $J_0(s)$ are marked as z^0_1, z^0_2, z^0_3, and z^0_4, and the first three zeros of $J_3(s)$ as z^3_1, z^3_2, and z^3_3.

(where dots represent derivatives with respect to s), the general solution is

$$R(r) = AJ_n(s) + BY_n(s) = AJ_n(\sqrt{\lambda}\, r) + BY_n(\sqrt{\lambda}\, r).$$

But $Y_n(r)$ is unbounded as $r \to 0$; thus the second boundary condition in (2.69b) requires that $B = 0$. The first then requires that $J_n(\sqrt{\lambda}\, a) = 0$. Now $J_n(s)$ has an infinite number of positive zeros, which we will denote by z^n_k, where $0 < z^n_1 < z^n_2 < \cdots$; Figure 2.3 shows plots of $J_0(s)$ and $J_3(s)$, with the zeros of each labeled. Thus we can achieve $J_n(\sqrt{\lambda}\, a) = 0$ precisely by taking λ to be one of the values $\lambda^n_k = (z^n_k/a)^2$. That is, for each n, $\lambda^n_1, \lambda^n_2, \ldots$ are the eigenvalues of the problem (2.69). For each n, k we will let $\phi^n_k(r)$ denote the corresponding eigenfunction. Thus for $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$,:

$$\lambda^n_k = \left(\frac{z^n_k}{a}\right)^2, \qquad \phi^n_k(r) = J_n\bigl(\sqrt{\lambda^n_k}\, r\bigr) = J_n\bigl(z^n_k r/a\bigr). \tag{2.70}$$

Summary: We have found the following solutions to our original PDE (2.56) which satisfy the boundary condition (2.57) (see (2.61), (2.66), (2.67), and (2.70)):

$$u(r,\theta,t) = \phi_k^0(r)e^{-(z_k^0\alpha/a)^2 t}, \qquad k = 1,2,3,\ldots, \tag{2.71a}$$

and for $n = 1,2,3,\ldots$,

$$u(r,\theta,t) = \phi_k^n(r)\cos(n\theta)e^{-(z_k^n\alpha/a)^2 t}, \qquad k = 1,2,3,\ldots \tag{2.71b}$$

and

$$u(r,\theta,t) = \phi_k^n(r)\sin(n\theta)e^{-(z_k^n\alpha/a)^2 t}, \qquad k = 1,2,3,\ldots \tag{2.71c}$$

In these formulas $\phi_k^n(r) = J_n\big(z_k^n r/a\big)$, with z_k^n the k^{th} zero of $J_n(s)$.

Finally we take up the problem of satisfying the initial condition (2.58). Since the PDE (2.56) and boundary condition (2.57) are linear and homogeneous, any linear combination of the solutions (2.71) found above will also satisfy (2.56) and (2.57). Thus we look for a solution of our problem in the form

$$u(r,\theta,t) = \sum_{k=1}^{\infty} c_k^0\,\phi_k^0(r)e^{-(z_k^0\alpha/a)^2 t} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\phi_k^n(r)\big[c_k^n\cos n\theta + d_k^n\sin n\theta\big]e^{-(z_k^n\alpha/a)^2 t}. \tag{2.72}$$

We must answer the crucial question: can we choose the constants c_k^n so that the solution (2.72) satisfies the initial condition, i.e., so that

$$f(r,\theta) = \sum_{k=1}^{\infty} c_k^0\,\phi_k^0(r) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\phi_k^n(r)\big[c_k^n\cos n\theta + d_k^n\sin n\theta\big], \tag{2.73}$$

and if the answer is yes, how do we compute the correct constants?

Remark 2.11: Recall how this question is answered for the one-dimensional regular Sturm-Liouville problem (see Remark 2.8): if the eigenfunctions of the problem are $\phi_n(x)$ then any function $f(x)$ defined for $a < x < b$ can be expanded as

$$f(r,\theta) = \sum_n c_n\phi_n(x); \qquad c_n = \frac{\langle f(x),\phi_n(x)\rangle_w}{\langle\phi_n(x),\phi_n(x)\rangle_w}, \tag{2.74}$$

where

$$\langle g(x),h(x)\rangle_w = \int_a^b g(x)h(x)w(x)\,dx. \tag{2.75}$$

Our text does not state general results for singular or periodic Sturm-Liouville problems, but in most cases, and in particular for the two problems (2.64) and (2.69) encountered in these notes, the same formulas apply.

To apply this remark to (2.73) we first consider $f(r,\theta)$ as a function of θ, for fixed r. The function $f(r,\theta)$ should of course be viewed as periodic in θ, with period 2π, again because angles differing by multiple of 2π represent the same physical point. As such a function it has a Fourier series expansion in terms of $\cos nx$ and $\sin nx$, but of course the coefficients will depend on the value of r. Thus we know that there is an expansion

$$f(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} \left[a_n(r)\cos n\theta + b_n(r)\sin n\theta\right], \tag{2.76}$$

with

$$\begin{aligned} a_0(r) &= \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,d\theta, \\ a_n(r) &= \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos n\theta\,d\theta, \quad n \ge 1, \\ b_n(r) &= \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin n\theta\,d\theta, \quad n \ge 1. \end{aligned} \tag{2.77}$$

Here we have referred to (2.76) as the usual Fourier series, but we can just as well think of it as the Sturm-Liouville series (2.74) associated with the periodic Sturm-Liouville problem (2.64).

Now we expand each of the functions $a_0(r)$, $a_n(r)$, and $b_n(r)$ in the Sturm-Liouville series (2.74) associated with the singular Sturm-Liouville problem (2.69). The eigenfunctions are the $\phi_k^n(r)$ given in (2.70); thus we have

$$a_0(r) = \sum_{k=1}^{\infty} c_k^0\phi_k^0(r), \quad a_n(r) = \sum_{k=1}^{\infty} c_k^n\phi_k^n(r), \quad b_n(r) = \sum_{k=1}^{\infty} d_k^n\phi_k^n(r), \qquad n \ge 1. \tag{2.78}$$

Inserting the formulas (2.78) into (2.76) yields precisely the desired expansion (2.73). The coefficients c_k^0, c_k^n, and d_k^n are given by the formula in (2.74). Note that here $w(r) = r$ (see Remark 2.10), so

$$c_k^0 = \frac{\displaystyle\int_0^a a_0(r)\phi_k^0(r)\,r\,dr}{\displaystyle\int_0^a \phi_k^0(r)^2 r\,dr}, \quad c_k^n = \frac{\displaystyle\int_0^a a_n(r)\phi_k^n(r)\,r\,dr}{\displaystyle\int_0^a \phi_k^n(r)^2 r\,dr}, \quad d_k^n = \frac{\displaystyle\int_0^a b_n(r)\phi_k^n(r)\,r\,dr}{\displaystyle\int_0^a \phi_k^n(r)^2 r\,dr}, \tag{2.79}$$

where as usual $n \ge 1$. Equations (2.76) and (2.78) answer the first part of the "crucial question" posed above affirmatively, while (2.77) and (2.79) give the formulas for the coefficients.

Finally, we want to note another way of looking at (2.79). Let us write ψ_k^n for the functions which appear in our solution (2.72):

$$\psi_k^0(r,\theta) = \phi_k^0(r), \quad \psi_{k,1}^n(r,\theta) = \phi_k^n(r)\cos n\theta, \quad \psi_{k,2}^n(r,\theta) = \phi_k^n(r)\sin n\theta, \quad n,k \ge 1.$$

Then with (2.77) the equations (2.79) become

$$c_k^0 = \frac{\int_0^a \int_0^{2\pi} f(r,\theta)\psi_k^0(r,\theta)\, r\, d\theta\, dr}{\int_0^a \int_0^{2\pi} \psi_k^0(r,\theta)^2\, r\, d\theta\, dr}, \tag{2.80a}$$

$$c_k^n = \frac{\int_0^a \int_0^{2\pi} f(r,\theta)\psi_{k,1}^n(r,\theta)\, r\, d\theta\, dr}{\int_0^a \int_0^{2\pi} \psi_{k,1}^n(r,\theta)^2\, r\, d\theta\, dr}, \quad d_k^n = \frac{\int_0^a \int_0^{2\pi} f(r,\theta)\psi_{k,2}^n(r,\theta)\, r\, d\theta\, dr}{\int_0^a \int_0^{2\pi} \psi_{k,2}^n(r,\theta)^2\, r\, d\theta\, dr}. \tag{2.80b}$$

But with (2.80) the expansion (2.73) is just a two-dimensional version of (2.74):

$$f(r,\theta) = \sum_{k=1}^\infty c_k^0\, \psi_k^0(r,r) + \sum_{k=1}^\infty \big[c_k^n \psi_{k,1}^n(r,\theta) + d_k^n \psi_{,2}^n k(r,\theta)\big],$$

$$c_k^0 = \frac{\langle f, \psi_k^0\rangle_2}{\langle \psi_k^0, \psi_k^0\rangle_2}, \quad c_k^n = \frac{\langle f, \psi_{k,1}^n\rangle_2}{\langle \psi_{k,1}^n, \psi_{k,1}^n\rangle_2}, \quad d_k^n = \frac{\langle f, \psi_{k,2}^n\rangle_2}{\langle \psi_{k,2}^n, \psi_{k,2}^n\rangle_2}, \quad n,k \ge 1,$$

where $\langle g(r,\theta), h(r,\theta)\rangle_2 = \int_0^a \int_0^{2\pi} g(r,\theta)h(r,\theta)\, r\, d\theta\, dr.$