Fractional range sine and cosine series

**Note:** In each of the four examples that follow—half range sine (HRS) series, half range cosine (HRC) series, quarter range sine (QRS) series, and quarter range cosine (QRC) series—the expansion given for a function $f(x)$ may be thought of as either

- the expansion of $f(x)$ on $[0, L]$ in terms of the given orthogonal set, or
- the Fourier series of the given periodic extension of $f(x)$. 
### Half range sine (HRS) series

**BV problem**

\[
X''(x) + \lambda X(x) = 0, \quad 0 < x < L \\
X(0) = 0, \quad X(L) = 0
\]

**Orthogonal set on \([0, L]\)**

\[
\{ \sin \frac{n\pi x}{L} \mid n = 1, 2, 3 \ldots \}
\]

**Expansion of**

\[f(x) = \sum_{n=1,2,3,\ldots} b_n \sin \frac{n\pi x}{L}\]

\[b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx\]

**Relevant periodic extension** \(f_1(x)\) of \(f(x) = x\) (figure drawn for \(L = \pi\))
## Half range cosine (HRC) series

### BV problem

\[
X''(x) + \lambda X(x) = 0, \quad 0 < x < L \\
X'(0) = 0, \quad X'(L) = 0
\]

### Orthogonal set on \([0, L]\)

\[
\{1\} \cup \left\{ \cos \frac{n\pi x}{L} \mid n = 1, 2, 3 \ldots \right\}
\]

### Expansion of \(f(x)\) in \(C_p[0, L]\)

\[
f(x) = a_0 + \sum_{n=1,2,3,\ldots} a_n \cos \frac{n\pi x}{L}
\]

\[
a_0 = \frac{2}{L} \int_{0}^{L} f(x) \, dx
\]

\[
a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx
\]

### Relevant periodic extension \(f_2(x)\) of \(f(x) = x\) (figure drawn for \(L = \pi\))
## Quarter range sine (QRS) series

**BV problem**

\[
X''(x) + \lambda X(x) = 0, \quad 0 < x < L
\]
\[
X(0) = 0, \quad X'(L) = 0
\]

**Orthogonal set on \([0, L]\)**

\[
\left\{ \sin \frac{n\pi x}{2L} \mid n = 1, 3, 5 \ldots \right\}
\]

**Expansion of**

\[
f(x) = \sum_{n=1,3,5,\ldots} b_n \cos \frac{n\pi x}{2L}
\]

\[
b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} \, dx
\]

**Relevant periodic extension \(f_3(x)\) of**

\(f(x) = x\) (figure drawn for \(L = \pi\))

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[Quarter Range Sine Extension of \(f(x) = x\) from \([0, \pi]\)]
**Quarter range cosine (QRC) series**

| BV problem | \(X''(x) + \lambda X(x) = 0, \quad 0 < x < L\)  
|            | \(X'(0) = 0, \quad X(L) = 0\)  |
| Orthogonal set on \([0, L]\) | \(\left\{\cos\frac{n\pi x}{2L} \mid n = 1, 3, 5\ldots\right\}\)  |
| Expansion of |  \(f(x) = \sum_{n=1,3,5,\ldots} a_n \cos\frac{n\pi x}{2L}\)  |
|  \(f(x) \in C_p[0, L]\) |  \(a_n = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{2L} \, dx\)  |
| Relevant periodic extension \(f_4(x)\) of  \(f(x) = x\) (figure drawn for \(L = \pi\)) |  ![Quarter Range Cosine Extension of \(f(x) = x\) from \([0, \pi]\)](image)  |