Various Fourier-type expansions:

We write $f(x) \sim$ series to indicate that the given series is some Fourier-type expansion of $f(x)$. All the formulas for coefficients come from the formula

$$c_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$$

with $\varphi_1, \varphi_2, \ldots$ a complete orthogonal set for $C_p[-\ell, \ell]$ or $C_p[0, L]$.

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right];$$

$$a_0 = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)\, dx, \quad a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx. \tag{1}$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}; \qquad c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\, dx. \tag{2}$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad (0 < x < L);$$

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \tag{3}$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (0 < x < L); \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{4}$$

$$f(x) \sim \sum_{n=1,3,5,\ldots} a_n \cos \frac{n\pi x}{2L} \quad (0 < x < L); \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{2L}\, dx. \tag{5}$$

$$f(x) \sim \sum_{n=1,3,5,\ldots} b_n \sin \frac{n\pi x}{2L} \quad (0 < x < L); \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L}\, dx. \tag{6}$$

Some antiderivatives:

$$\int e^{-ax} \sin(bx)\, dx = -\frac{e^{-ax}}{a^2 + b^2} \left(a \sin(bx) + b \cos(bx) \right) + C$$

$$\int e^{-ax} \cos(bx)\, dx = \frac{e^{-ax}}{a^2 + b^2} \left(-a \cos(bx) + b \sin(bx) \right) + C$$

$$\int x \cos(bx)\, dx = \frac{\cos(bx)}{b^2} + \frac{x \sin(bx)}{b} + C \qquad \int x \sin(bx)\, dx = \frac{\sin(bx)}{b^2} - \frac{x \cos(bx)}{b} + C$$

Some trig identities:

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \qquad\qquad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos(A)\cos(B) = (1/2)\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin(A)\cos(B) = (1/2)\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin(A)\sin(B) = (1/2)\left[\cos(A-B) - \cos(A+B)\right]$$

Sturm-Liouville problem:

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \qquad \langle f(x), y(x) \rangle_w = \int_0^L f(x) g(x) w(x)\, dx$$