(RECALL)

STORM-LIOUVILLE THEORY

(ODE) \( (p(x)y')' + q(x)y + \lambda\omega(x)y = 0 \), \( a < x < b \)

\( p, p', q, \omega \) are all continuous \( a \leq x \leq b \)
and \( p(x), \omega(x) \) are strictly positive ie \( > 0 \).

\{ Regular S-L Problem. \}

If some of these conditions don’t hold, \( \Rightarrow \) SINGULAR S-L Problem.

\( \) Boundary cond. : \( y(a) + \beta y'(a) = 0 \) & \( y(b) + \gamma y'(b) = 0 \)

SEPARATED Boundary cond. (two ends are separate).

We may have Boundary Conditions which are not separated.

Ex: Periodic BC : \( y(a) = y(b) \) \( y'(a) = y'(b) \)

(SECTION 17.4)

SINGULAR S-L PROBLEMS or NON-SEPARATED BOUND. COND.

For Singular S-L problems or non-separated Boundary Conditions, part of our theorem can fail.

Example: Heat Conduction in a circular disk. (See Notes on this on webpage)

disc: radius ‘a’
\( u(x, y, t) \) : temperature

Heat equation in 2-D:
\( u_t = \alpha^2 (u_{xx} + u_{yy}) = \alpha^2 \Delta u = \alpha^2 \nabla^2 u \)
(as laplace eqn form)
Go to polar co-ordinates, \( u(r, \theta, t) \)

PDE: 
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}
\]

To get this, start with
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha^2} \frac{\partial u}{\partial t}
\]
Change
\[
\frac{\partial}{\partial x} \rightarrow \frac{1}{r} \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial y} \rightarrow \frac{1}{r} \frac{\partial}{\partial \theta}
\] using chain rule.

\( u(r, \theta, 0) = f(r, \theta) \) (some function of \( r, \theta \))

\( u(a, \theta, t) = 0 \quad \forall \theta, t \geq 0 \) (holding the edge of disc at zero temp)

1. Look for product solutions of PDE.
2. Impose BC\( \rightarrow \) eigenvalues

Look for solutions \( u(r, \theta, t) = R(r) \Theta(\theta) T(t) \)

\[
\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{T}{T} \quad \text{(dividing this eqn by } R \Theta T)
\]

\[
\Rightarrow R \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = -\lambda
\]

depends only on \( R, \Theta \), \( \frac{1}{\alpha^2} \) only on \( T \) \( \Rightarrow \) only possible when both sides are constant (we will say \( -\lambda : \text{say} \))

\( \Rightarrow \)

\( T = -a^2 T \) \( \Rightarrow T(t) = e^{-a^2 t} \) \( \text{from a constant here, but we add the constants while doing superposition of soln.} \)

\( \Rightarrow \) It’s ok even if we put const. now.

\[
\Rightarrow \frac{1}{R} \frac{\partial^2 R}{\partial r^2} + \frac{\lambda}{R} = \Theta'' = \Theta'' + \Theta'' = -\Theta'' = \lambda \Theta + \lambda (\Theta')' \Rightarrow \text{again this}
\]

\[
\Rightarrow (\Theta'' + \mu \Theta = 0 \quad \text{(ODE in } \Theta) \quad 0 \leq \Theta \leq 2\pi \quad \text{but BC?}
\]

\[
\text{periodic boundary condn.} \quad \Rightarrow \text{same for } \Theta \quad \text{equal at } \Theta = 0, 2\pi \text{ as physically they are the same point.}
\]

\( \Theta(0) = \Theta(2\pi) \quad \Theta(0) = \Theta(2\pi) \)

\( \rightarrow \) This is a S-L problem w/ periodic BC!
\[ \mu < 0 : \text{Not possible} \quad \mu = -k^2 \]

\[ \Theta = A \cosh kx + B \sinh kx \quad \Theta' = kA \sinh kx + kB \cosh kx \]

\[ \Theta(0) = \Theta(2\pi) \Rightarrow A = A \cosh 2\pi k + B \sinh 2\pi k \]

\[ \Theta'(0) = \Theta'(2\pi) \Rightarrow kB = kA \sinh 2\pi k + kB \cosh 2\pi k \quad \text{we know } k \neq 0 \]

\[ B = A \sinh 2\pi k + B \cosh 2\pi k \]

\[ (-\cosh 2\pi k) A - (\sinh 2\pi k) B = 0 \]

\[ (-\sinh 2\pi k) B + (1 - \cosh 2\pi k) B = 0 \quad \text{set of homogeneous linear equations} \]

\[ \text{determinant has to be zero} \quad (\text{only possible soln}: A = B = 0) \]

\[ 1 - \cosh 2\pi k \sinh 2\pi k = 1 + \cosh^2 2\pi k - 2 \cosh 2\pi k \sinh^2 2\pi k \]

\[ \cosh^2 x - \sinh^2 x = 1 \]

\[ = 2(1 - \cosh 2\pi k) = 0 \Rightarrow \cosh 2\pi k = 1 \]

\[ \text{which is possible for } k^2 = 0 \]

\[ \mu = -k^2 < 0 \quad \Rightarrow \text{not possible} \]

\[ \mu > 0 : \mu = k^2 \]

\[ \Theta = A \cos kx + B \sin kx \quad (\text{solving like above}) \quad \cos 2\pi k = 1 \]

\[ \Rightarrow k = n = 1, 2, 3, \ldots \]

\[ \Rightarrow \text{Eigenvalues: } \mu_n = n^2 ; \quad n = 1, 2, 3 \]

\[ \text{Periodic/boundary condn still same, gives} \]

\[ (1 - \cos 2\pi k) A - (\sin 2\pi k) B = 0 \]

\[ (\sin 2\pi k) A + (1 - \cos 2\pi k) B = 0 \]

\[ \text{for } k = n \Rightarrow A \cdot 0 - B = 0 \quad ; \quad A + 0 \cdot B = 0 \]

\[ \text{which is true for all } A, B. \]

\[ \Rightarrow \text{For } \mu_n = n^2, \text{ we have two eigenfunctions!! } \]

\[ \Theta_{n,1} = \sin n\theta \quad \Theta_{n,2} = \cos n\theta \] - (2)

\[ \mu = 0: \Theta = A + B \theta \]

\[ \Theta(0) = \Theta(2\pi) \Rightarrow B = 0 \quad \Theta'(0) = \Theta'(2\pi) = 0 \Rightarrow A \text{ can be anything} \]

\[ \text{[constant soln] satisfies both BC/IC} \quad \Rightarrow \text{soln is any const. say} = C \]

\[ \mu = 0 \quad B = 1 - (3) \]
Differentiate:

\[
\frac{d}{dx} (x^2 R') - x^2 R + \lambda^2 R = 0 \quad 0 \leq x \leq a
\]

(divide by \(x\))

\[
\frac{d}{dx} (x^2 R') - x^2 R + \lambda^2 R = 0
\]

S-L problem:

\[(p y')' + q y + \lambda w y = 0\]

\[
p(x) = x \quad q(x) = -\frac{x^2}{2} \quad w(x) = x
\]

But \(p(0) = w(0) = 0\) \& \(q(0) \to \infty \) as \(x \to 0 \Rightarrow \text{S-L Problem}

S-L Theorem's conditions that \(p, w\) are strictly positive fails.

Also, \(q\) is not continuous at 0 w/c is in range 0 \(\leq L \leq a\).

\(BC: R(a) = 0 \) (from \(u(x, \theta, t) = 0\))

\(x = 0 \) is a pt. \(\Rightarrow \) No real BC as before at \(x = 0\).

As we can see from above, \(p, w\) vanish at \(x = 0\) \& \(q \to \infty\)

\(\Rightarrow x = 0 \) is a singular point.

Require \(R\) be finite as \(x \to 0\) \[since we don't want \(\text{temp.} \to \infty \) \]

\[
\frac{d}{dx} (x^2 R') - x^2 R + \lambda^2 R = 0 \Rightarrow \lambda^2 R'' + x^2 R' + (\lambda^2 - n^2)R = 0
\]

\(\Rightarrow\) New variable: \(s = \sqrt{\lambda} x \) \(\lambda\) is positive

\[
\frac{d^2 R}{ds^2} + s \frac{dR}{ds} + (s^2 - n^2) R = 0 \Rightarrow \text{Bessel's equation of order } n!
\]

(we know from before) \(R(x) = A J_n(\sqrt{\lambda} x) + B Y_n(\sqrt{\lambda} x)\) \(\text{(see lecture on)}\)

Now, \(y_n \to \infty\) as \(x \to 0\) \(\Rightarrow \) BUT \(R(x)\) has to be finite \(\Rightarrow \) \(B = 0\).

\(\Rightarrow R(x) = J_n(\sqrt{\lambda} x)\)

\(BC: R(a) = 0\)

\(Z_k^n: K^{th}\) positive zero of \(J_n\)

\[Z_n^1, Z_n^2, Z_n^3, \ldots\]

\(\Rightarrow \text{MUST Have } \sqrt{\lambda} a = Z_k^n \text{ for some } K \text{ for } J_n(\sqrt{\lambda} a) = 0\)

\[
\lambda_k^n = \left(\frac{Z_k^n}{a}\right)^2
\]

\(n\) = which Bessel's fun we're looking at

\(K = \text{we eigenvalue of that Bessel fun}\)

Eigenfunction: \(\phi_k^n(x) = J_n\left(\frac{Z_k^n}{a} x\right)\)
Product Solutions:

\[ n = 0 \quad \phi_k^0(x) e^{-\alpha^2 \lambda_k t} \quad \text{(for } n = 0 \text{, } \Theta \text{ = constant value,) } \quad k = 1, 2, 3, \ldots \]

(we're a family of eigenfunctions for } n = 0 \text{ !)

\[ n > 0 \quad \phi_k^n(x) \cos(n \Theta) e^{-\alpha^2 \lambda_k t} \quad \phi_k^n(x) \sin(n \Theta) e^{-\alpha^2 \lambda_k t} \quad k = 1, 2, 3, \ldots \]

General Solution:

\[ u(x, \Theta, t) = \sum_{k=1}^{\infty} c_k \phi_k^0(x) e^{-\alpha^2 \lambda_k t} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \phi_k^n(x) (c_k \cos(n \Theta) + d_k \sin(n \Theta)) e^{-\alpha^2 \lambda_k t} \]

Initial Condition: \[ u(x, \Theta, 0) = \sum_{k=1}^{\infty} c_k \phi_k^0(x) + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \phi_k^n(x) (c_k \cos(n \Theta) + d_k \sin(n \Theta)) \]