EXAM 12/19 - QUESTION SESSION 12/11 AFTERNOON

Office Hours 12/18

WAVE EQUATION ON AN INTERVAL

\[ c^2 U_{xx} = U_t \quad 0 \leq x \leq L, \quad t > 0 \]

\[ U(0,t) = U(L,t) = 0 \quad U(x,0) = f(x) \quad U_t(x,0) = g(x) \]

\[ U(x,t) = \sum_{n=1}^{\infty} \left[ R_n \cos \frac{nx \pi}{L} + S_n \sin \frac{nx \pi}{L} \right] \]

\[ R_n = \frac{2}{L} \int_0^L f(x) \sin \frac{nx \pi}{L} \, dx \]

\[ S_n = \frac{2}{L} \int_0^L g(x) \sin \frac{nx \pi}{L} \, dx \]

REMARKS:

1. THIS IS A SUPERPOSITION OF VIBRATIONS WITH DIFFERENT FREQUENCIES

\[ \omega_n = \frac{nx \pi}{L} \quad \omega_1: \text{FUNDAMENTAL FREQUENCY} \]

\[ \omega_n: n \geq 2 \quad \text{OVERTONES} \]

2. REWRITE THE SOLUTION

\[ \sin A \cos B = \frac{1}{2} \left[ \sin (A+B) + \sin (A-B) \right] \]

\[ \sum_{n=1}^{\infty} R_n \sin \frac{nx \pi}{L} \cos \frac{nx \pi}{L} = \frac{1}{2} \sum_{n=1}^{\infty} R_n \left[ \sin \frac{nx \pi}{L} (x+c) + \sin \frac{nx \pi}{L} (x-c) \right] \]

HPS.

\[ f(x) = \sum_{n=1}^{\infty} R_n \sin \frac{nx \pi}{L} \quad 0 \leq x \leq L \]

FOR ALL \( x \):

\[ f_1(x) = \sum_{n=1}^{\infty} R_n \sin \frac{nx \pi}{L} \rightarrow f_1: \text{EXTENSION OF} \ f \]

TO ODD FUNCTION, PERIOD 2L

\[ f_1(x) \]
(1) Becomes

\[ \frac{1}{2} \left[ f_i(x + ct) + f_i(x - ct) \right] \]

\( f_i(x + ct) \) Travelling Wave

\[
\begin{align*}
- t &= 0 \\
- t_i &= 0 \\
C \cdot \text{Velocity}
\end{align*}
\]

2nd Part: \( \sum_{n} S_n \sin \frac{n \pi x}{L} \sin \frac{n \pi c t}{L} \)

\[
S_n \sin \frac{n \pi c t}{L} = \left[ \cos(n \pi (A - B)) - \cos(n \pi (A + B)) \right]^{1/2}
\]

\[
= \frac{1}{2} \sum_{n} \left[ \cos \frac{n \pi}{L} (x - ct) - \cos \frac{n \pi}{L} (x + ct) \right]
\]

If \( h(x) = \frac{1}{n} S_n \cos \frac{n \pi x}{L} \)

\[
= \frac{1}{2} \left[ h(x - ct) - h(x + ct) \right]
\]

Can Yield \( \int_{x-ct}^{x+ct} g_i(z) \, dz \)

So

\[
\begin{align*}
U(x, t) &= \frac{1}{2} \left( f_i(x - ct) + h(x - ct) \right) \\
&\quad + \frac{1}{2} \left( f_i(x + ct) - h_i(x + ct) \right) \\
&= \frac{1}{2} \left( f_i(x - ct) + f_i(x + ct) \right) + \frac{1}{2} \sum_{n} \int_{x-ct}^{x+ct} g_i(z) \, dz
\end{align*}
\]

- Shows Maple File of Wave Animations

3. \( c^2 \Delta X = U_{tt} \rightarrow U(x, t) = X(x)T(t) \)

\[
\begin{align*}
X''(x) &= 0, & T'' + c^2 \lambda T &= 0 \\
X(0) &= X(L) = 0 \\
X(0) &= X(L) = 0
\end{align*}
\]

For \( \lambda = 0 \):
\[ x(x) = Ax + B \quad \lambda = 0 \]
\[ u(x) = (Ax+B)(C+D) = \alpha + \beta x + \gamma + 5x^2 + \text{ THIS CAN REPRESENT A STEADY STATE:} \]
\[ \text{BC: } u(0,+) = u_1, \quad u(L+) = u_2 \]
\[ \text{STEADY STATE: } u(x) = u + (u_2 - u_1) \frac{x^2}{2} \]
\[ u(x+) = v(x) + w(x+) \]

4. Can easily consider other BC

Neumann: \[ u_x(0,+) = 0 \]

Physically: PHYSICALLY: STRAIGHTFORWARD

\[ \text{WAVE EQUATION ON LINE } \]
\[ c^2 u_{xx} = u_{tt} \]
\[ u(x,0) = f(x) \]
\[ u_t(x,0) = g(x) \]

Want \( f, g \) vanishing as \( x \to \infty \)

DALEMBERT'S SOLUTION

INTRODUCE NEW VARIABLES
\[ \xi = x - ct \]
\[ \eta = x + ct \]

REWRITE EQUATION
\[ c^2 U_{xx} = U_{tt} \quad \Rightarrow \quad \text{Look for } \ U(3, t) = ? \]

\[ \frac{3}{2} \frac{\partial^2 U}{\partial x^2} \frac{\partial^2 U}{\partial x^2} + \frac{2}{2} \frac{\partial^2 U}{\partial x \partial t} \frac{\partial^2 U}{\partial x \partial t} = \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} \]

\[ \frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial x \partial t} \]

\[ U_{xx} = \frac{3}{2} \left( \frac{2}{2} U_{xx} \right) \left( \frac{2}{2} \frac{\partial^2 U}{\partial x \partial t} + \frac{\partial^2 U}{\partial x \partial t} \right) \]

\[ = U_{xx} + 2 U_{xx} + U_{tt} \]

\[ U_{tt} = \left( -\frac{3}{2} + \frac{3}{2} \right) \left( -\frac{3}{2} + \frac{3}{2} \right) \]

\[ = c^2 \left[ U_{xx} - 2 U_{xx} + U_{tt} \right] \]

\[ c^2 U_{xx} = U_{tt} \]

\[ c^2 \left[ U_{xx} + 2 U_{tt} = U_{tt} \right] = c^2 \left[ U_{xx} - 2 U_{tt} + U_{tt} \right] \]

\[ 4c^2 U_{tt} = 0 \Rightarrow \quad \boxed{U_{tt} = 0} \]

**Solve:**

\[ \frac{2}{2} \left[ U_{xx} \right] = 0 \]

\[ \Rightarrow \quad U_{xx} = H(3) \]

\[ U = \int H(3) \, d\xi = F(3) + G(\eta) \]

\[ U(3, \eta) = F(3) + G(\eta) \]

\[ \Rightarrow \]

\[ U(x, t) = F(x - c t) + G(x + c t) \]

\[ F \text{ travels to the left} \]

\[ G \text{ travels to the right} \]

\[ F, G \text{ are arbitrary} \]

**Initial Conditions:**

\[ U(x, 0) = f(x) = F(x) + G(x) \]

\[ U(x, 0) = g(x) = -cF'(x) + cG'(x) \]

\[ U(x, t) = -cF'(x - c t) + cG'(x + c t) \]
INTEGRATE: \( h(x) = \int g(x) \, dx + C \)
\[ = \int_0^x g(y) \, dy \]
\[-cF(x) + cG(x) = h(x) + K \quad \text{(1)} \]
\[ F(x) + G(x) = f(x) \quad \text{-- multiply by } c \text{ and add} \]
with \( \text{Eqn (1)} \)
\[ 2cG(x) = h(x) + cf(x) + K \]
\[ G(x) = \frac{h(x)}{2c} + \frac{f(x)}{2} + \frac{K}{2c} \]
\[ F(x) = \frac{f(x)}{2} - \frac{h(x)}{2c} - \frac{K}{2c} \]

\[ u(x,t) = F(x-c+t) + G(x+c-t) \]
\[ = \frac{1}{2} \left[ f(x-c+t) + f(x+c-t) \right] + \frac{1}{2c} \left[ h(x+c-t) - h(x-c+t) \right] \]

\[ u(x,t) = \frac{1}{2} \left[ f(x-c+t) + f(x+c-t) \right] + \frac{1}{2c} \int_{x-c-t}^{x+c-t} g(y) \, dy \]

SHOWS MAPLE FILE W/ ANIMATIONS

\[ x = c(t+A) \]
\[ (x,+t) \]
\[ x = -c(t+B) \]

KNOW \( g(y) \) FOR \( x-c \leq y \leq x+c \)

--- DOMAIN OF DEPENDENCE OF \( u(x,t) \)

SOMETIMES

--- DOMAIN OF DEPENDENCE IF (DIFFERENT PROBLEM) EXTERNAL INFLUENCE AT TIMES \( x,t \)

\[ c^2 U_{xx} - U_{tt} = h(x,t) \]
--- KNOWN
WAVE EQUATION: INFLUENCES CANNOT TRAVEL FASTER THAN SPEED C.

Book uses term "DOMAIN OF INFLUENCE"

BUT....

DOMAIN OF INFLUENCE
OF FIGURE FOR
\[ a \leq x \leq b \]

\[ a \]
\[ b \]

LINES \[ x = \pm c + t \] ARE CALLED CHARACTERISTICS OF THE WAVE EQUATION