VINESH NISHAWALA

12-04-08

EXAM 12/19 - QUESTION SESSION 12/11 AFTERNOON

OFFICE HOURS 12/18

WAVE EQUATION ON AN INTERVAL

$$c^2 u_{xx} = u_{tt} \qquad 0 \le x \le L, \quad t \ge 0$$

$$u(0,t) = u(L,t) = 0 \quad u(x,0) = f(x) \quad u_t(x,0) = g(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left[R_n \cos\frac{n\pi c t}{L} + S_n \sin\frac{n\pi c t}{L}\right]$$

$$R_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$
$$S_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

REMARKS:

① THIS IS A SUPERPOSITION OF VIBRATIONS WITH DIFFERENT FREQUENCIES

$\omega_n = \frac{n\pi c}{L}$ $\quad \omega_1$: FUNDAMENTAL FREQUENCY

ω_n: $n \ge 2$: OVERTONES

② REWRITE THE SOLUTION:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

So

$$\sum_{n=1}^{\infty} R_n \sin\frac{n\pi x}{L}\cos\frac{n\pi c t}{L} = \frac{1}{2}\sum_{n=1}^{\infty} R_n\left[\sin\frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)\right] \quad (*)$$

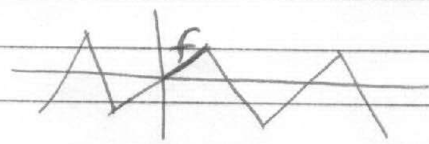
HRS.

$$f(x) = \sum_{n=1}^{\infty} R_n \sin\frac{n\pi x}{L} \qquad 0 \le x \le L$$

FOR ALL x:

$$f_1(x) = \sum_{n=1}^{\infty} R_n \sin\frac{n\pi x}{L} \rightarrow f_1\text{: EXTENSION OF } f$$

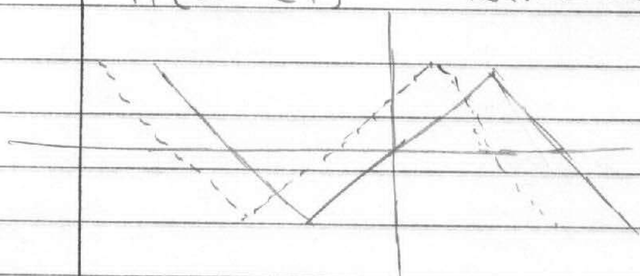
TO ODD FUNCTION, PERIOD 2L



(*) BECOMES

$$\tfrac{1}{2}\left[f_1(x+ct)+f_1(x-ct)\right]$$

$f_1(x+ct)$ TRAVELLING WAVE



— $t=0$

---- $t_1>0$

c: VELOCITY

<u>2ND PART</u>: $\sum S_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}$

$$\sin A\sin B = \left[\cos(A-B)-\cos(A+B)\right]\tfrac{1}{2}$$

$$=\tfrac{1}{2}\sum S_n\left[\cos\tfrac{n\pi}{L}(x-ct)-\cos\tfrac{n\pi}{L}(x+ct)\right]$$

IF $h(x)=\sum_n S_n\cos\frac{n\pi}{L}x$

$$=\tfrac{1}{2}\left[h(x-ct)-h(x+ct)\right]$$

CAN YIELD $\Rightarrow$ $\frac{1}{2c}\int_{x-ct}^{x+ct} g_1(z)\,dz$

<u>SO</u>

$$u(x,t)=\tfrac{1}{2}\left(f_1(x-ct)+h_1(x-ct)\right)+\tfrac{1}{2}\left(f_1(x+ct)-h_1(x+ct)\right)$$

$$=\tfrac{1}{2}\left(f_1(x-ct)+f_1(x+ct)\right)+\tfrac{1}{2c}\int_{x-ct}^{x+ct} g_1(z)\,dz$$

— SHOWS MAPLE FILE OF WAVE ANIMATIONS —

③ $c^2u_{xx}=u_{tt}\rightarrow u(x,t)=X(x)T(t)$

$X''+\lambda X=0,\quad T''+c^2\lambda T=0$

$X(0)=X(L)=0$

FOR $\lambda=0$:

EQUATION:

$X(x) = Ax + B \qquad \underline{\lambda = 0}$

$T(t) = Ct + D$

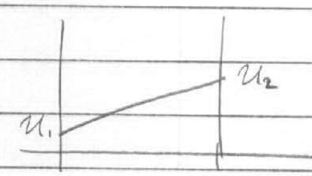
$u(x,t) = (Ax+B)(Ct+D) = \alpha + \beta x + \gamma t + \delta x t$

THIS CAN REPRESENT A STEADY STATE:

BC: $u(0,t) = u_1 \qquad u(L,t) = u_2$

STEADY STATE:

$$v(x) = u + (u_2 - u_1)\frac{x}{L}$$

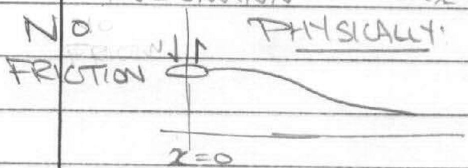


$$u(x,t) = v(x) + w(x,t)$$

④ CAN EASILY CONSIDER OTHER BC

NEUMANN: $u_x(0,t) = 0$

PHYSICALLY:



MATHMATICALLY:

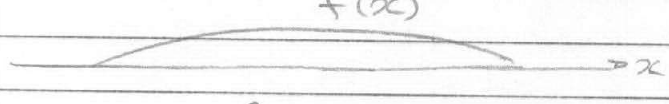
STRAIGHTFORWARD

WAVE EQUATION ON LINE

$c^2 u_{xx} = u_{tt} \qquad u(x,0) = f(x)$

$\qquad\qquad\qquad\quad u_t(x,0) = g(x)$

$f(x)$



WANT f, g VANISHING AS $x \to \pm\infty$

d'ALEMBERT'S SOLUTION

INTRODUCE NEW VARIABLES

$\xi = x - ct \qquad$ REWRITE EQUATION

$\eta = x + ct$

$c^2 u_{xx} = u_{tt} \longrightarrow$ LOOK FOR $u(\xi,\eta)=?$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -c\frac{\partial}{\partial \xi}+c\frac{\partial}{\partial \eta}$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u\right) = \left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\right)$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{tt} = \left(-c\frac{\partial}{\partial \xi}+c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial u}{\partial \xi}+c\frac{\partial u}{\partial \eta}\right)$$

$$= c^2\left[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right]$$

$$c^2 u_{xx} = u_{tt}$$

$$c^2\left[\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}\right] = c^2\left[\cancel{u_{\xi\xi}} + -2u_{\xi\eta} + \cancel{u_{\eta\eta}}\right]$$

$$4c^2 u_{\xi\eta} = 0 \Rightarrow \boxed{u_{\xi\eta}=0}$$

SOLVE:

$$\frac{\partial}{\partial \eta}\left[u_\xi\right] = 0$$

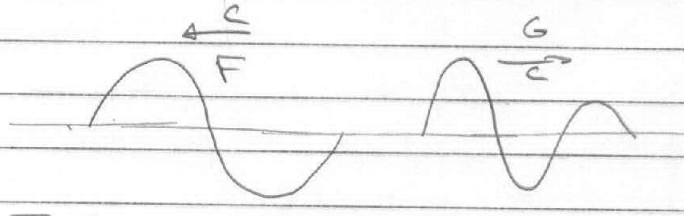
so $u_\xi = H(\xi)$

$$u = \int H(\xi)\,d\xi = F(\xi) + G(\eta)$$

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

OR

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)}$$



F TRAVELS TO THE LEFT

G TRAVELS TO THE RIGHT.

F, G ARE ARBITRARY.

INITIAL CONDITIONS:

$$u(x,0) = f(x) = F(x) + G(x)$$

$$u_t(x,0) = g(x) = -cF'(x) + cG'(x)$$

$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct)$$

INTEGRATE: LET $h(x) = \int g(x)\,dx + c$
$= \int_0^x g(y)\,dy$

$-cF(x) + cG(x) = h(x) + K$ ①

$F(x) + G(x) = f(x)$ → MULTIPLY BY C & ADD W/ EQN ①

$2cG(x) = h(x) + cf(x) + K$

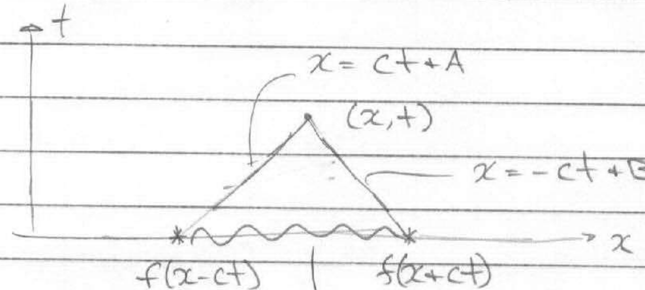
$G(x) = \frac{h(x)}{2c} + \frac{f(x)}{2} + \frac{K}{2c}$ $\quad G + F = f$

$F(x) = \frac{f(x)}{2} - \frac{h(x)}{2c} - \frac{K}{2c}$

$u(x,t) = F(x-ct) + G(x+ct)$
$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\left[h(x+ct) - h(x-ct)\right]$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy}$$

— SHOWS MAPLE FILE W/ ANIMATIONS —



→ KNOW $g(y)$ FOR $x-ct \le y \le x+ct$

〰 → DOMAIN OF DEPENDENCE OF $u(x,t)$

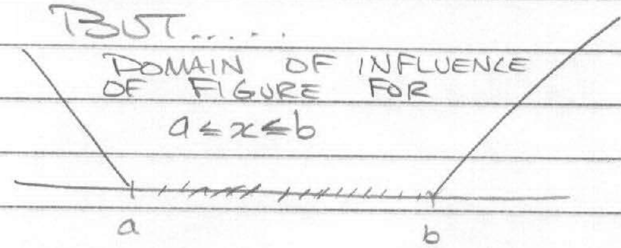
SOMETIMES

→ DOMAIN OF DEPENDENCE IF (DIFFERENT PROBLEM) EXTERNAL INFLUENCE AT TIMES x, t

$c^2 u_{xx} - u_{tt} = h(x,t)$
→ KNOWN

<u>WAVE EQUATION</u>: INFLUENCES CANNOT TRAVEL FASTER THAN SPEED C.

BOOK USES TERM "DOMAIN OF INFLUENCE" BUT.....



LINES $x = \pm ct + A$ ARE CALLED <u>CHARACTERISTICS</u> OF THE WAVE EQUATION