VINESH NISHAWALA

09-11-08

METHOD OF FROBENIUS

$$y'' + p(x)y' + q(x)y = 0$$

$x = x_0$ IS A RSP

SOLUTIONS $y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)$ $\quad a_0 \neq 0$

$r = r_1, r_2 \quad , \quad r_1 > r_2$

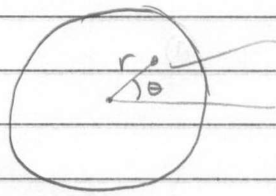
$$y_1(x) = (x-x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad a_0 \neq 0$$

$$y_2(x) = \begin{cases} & r_1 = r_2 \\ \text{3 CASES.} & r_1 - r_2 = m \\ & \text{OTHER} \end{cases}$$

APPLICATION: BESSEL FUNCTIONS.

MOTIVATION: WHY?

PROBLEM: HEAT CONDUCTION IN A DISC



(r, θ) $\quad u(r,\theta,t)$: TEMPERATURE AT POINT (r,θ), AT TIME t

HEAT EQUATION: $\boxed{\frac{1}{r}\frac{\delta}{\delta r}\left(r\frac{\delta u}{\delta r}\right) + \frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2} = \frac{1}{\alpha^2}\frac{\delta u}{\delta t}}$

α^2: CONDUCTIVITY OF DISC

IDEA: SEPARATION OF VARIABLES.

LOOK FOR SOLUTIONS IN THE SPECIAL FORM $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

PLUG IN: GET ORDINARY DIFFERENTIAL EQUATIONS FOR R, Θ, T

$T'(t) = -\lambda\alpha^2 T$ $\quad \lambda$: NEW CONSTANT

$\longrightarrow T(t) = Ae^{-\lambda\alpha^2 t}$

$\Theta''(\theta) = -n^2\,\Theta$ $\quad n$: CONSTANT INTEGER

$\Theta(\theta) = B\cos n\theta + C\sin n\theta$

$[\theta \to \theta + 2\pi, \ \Theta(\theta) \to \Theta(\theta)]$

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0$$ (MORE OR LESS, BESSEL'S EQN)

TO BE PRECISE: CHANGE VARIABLES

$$x = \sqrt{\lambda}\, r; \quad R(r) = y(x)$$

$$\boxed{x^2y'' + xy' + (x^2 - n^2)y = 0} \longrightarrow \boxed{\text{BESSEL'S EQN OF ORDER } n}$$

PRELIMINARY: GAMMA FUNCTION

FOR $x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \longrightarrow$$ DOES THIS CONVERGE?

WRITE

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t}\,dt + \int_1^\infty t^{x-1}e^{-t}\,dt$$

FIRST

$\int_0^1 t^{x-1}e^{-t}\,dt$ SINCE $e^{-t} \le 1$

$$\int_0^1 t^{x-1}e^{-t}\,dt \le \int_0^1 t^{x-1}\,dt \quad \boxed{x-1 > -1}$$

$$\int_0^1 t^p\,dt < \infty \text{ IF } \boxed{p > -1}$$

IF $p > -1 \to p+1 > 0$

$$\int_0^1 t^p\,dt = \left.\frac{t^{p+1}}{p+1}\right|_0^1 = \frac{1}{p+1}$$

SECOND

$\int_1^\infty t^{x-1}e^{-t}\,dt$ $\quad t^{x-1} \to$ GETS BIGGER

$e^{-t} \to$ GETS SMALLER

EXPONENTIAL ALWAYS WINS!

SUPPOSE $n > x$

$$e^{t} = \sum_{k=0}^{\infty} \frac{1}{k!} t^{k} \geq \frac{1}{n!} t^{n}$$

$$e^{-t} = 1/e^{t} \leq n!/t^{n}$$

$$\int_{1}^{\infty} t^{x-1} e^{-t}\, dt \leq n! \int_{0}^{\infty} t^{x-n-1}\, dt \rightarrow -(x-n-1) = 1 + (n-x) > 1$$

$$\int_{1}^{\infty} t^{-p}\, dt < \infty \quad \text{IF} \quad p > 1$$

PROPERTIES OF $\Gamma(x)$: $\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t}\, dt$

$$\Gamma(1) = \int_{0}^{\infty} e^{-t}\, dt = -e^{-t}\Big|_{0}^{\infty} = 1$$

$$\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t}\, dt \rightarrow \text{INTEGRATION BY PARTS.}$$

$$= -t^{x-1} e^{-t} + (x-1)\int_{0}^{\infty} t^{x-2} e^{-t}\, dt$$

$u = t^{x-1}$ $\quad v = -e^{-t}$

$du = (x-1)t^{x-2}$ $\quad dv = e^{-t}dt$

$\int u\,dv = uv - \int v\,du$

FOR $x > 1$

$$= (x-1)\int_{0}^{\infty} t^{x-2} e^{-t}\, dt \qquad x-2 \rightarrow (x-1)-1$$

$$= (x-1)\Gamma(x-1)$$

$$\boxed{\Gamma(x) = (x-1)\Gamma(x-1)} \quad \underline{x > 1}$$

$\Gamma(2) = 1 \cdot 1$

$\Gamma(3) = 2 \cdot 1$

$\Gamma(4) = 3 \cdot 2 \cdot 1$

$\Gamma(5) = 4 \cdot 3 \cdot 2 \cdot 1$

$$\boxed{\Gamma(n) = (n-1)!} \quad \underline{n > 0}$$

TAKE x TO BE BIG.

$$\Gamma(x) = (x-1)\Gamma(x-1) = (x-1)(x-2)\Gamma(x-2) = \cdots$$

$$\underline{x \to x+1}$$

$$\Gamma(x+1) = x(x-1)\cdots(x-k)\Gamma(x-k)$$

$$\boxed{x(x-1)\cdots(x-k) = \frac{\Gamma(x+1)}{\Gamma(x-k)}}$$

BESSEL FUNCTIONS.

BESSEL'S EQUATION OF ORDER ν "NU"

$$x^2y'' + xy' + (x^2-\nu^2)y = 0$$

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

$x=0$ IS A RSP

$$x\,p(x) = x\frac{1}{x} = 1 \quad \text{ANALYTIC}$$

$$x^2 q(x) = x^2\left(1-\frac{\nu^2}{x^2}\right) = x^2-\nu^2 \cdot \quad \text{ANALYTIC}$$

$\left.\right\} R=\infty$

SOLUTION WILL BE A SERIES W/ $R=\infty$ ALSO.

INDICIAL EQUATION

$$\gamma(r) = 0$$

$$\gamma(r) = r(r-1) + p_0 r + q_0$$

$$= r(r-1) + r - \nu^2$$

$$= r^2 - \nu^2$$

$$r = \pm\nu \quad (\nu > 0)$$

$$r_1 = \nu \quad r_2 = -\nu$$

TROUBLE: $r_1 = r_2$ $\boxed{\nu = 0}$

$r_1 - r_2 = m \to$ WHEN $\nu = k$ OR $\nu = \frac{2k+1}{2}$

FIND ONE SOLUTION $y_1(x)$ FOR $r = \nu$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\nu}$$

PLUG IN: RECURSION

$$\boxed{\gamma(n+\nu)\, a_n = -a_{n-2}} \quad n = 1, 2, 3, \dots \quad a_0 \neq 0$$

$n=1$ $\quad -a_1 = -\cancelto{0}{a_{-1}}$

$$\boxed{a_1 = 0} \quad \text{CLEARLY } a_1 = a_3 = a_5 = \dots = 0$$

$$a_{2k} = -\frac{a_{2(k-1)}}{\gamma(2k+\nu)} \qquad \gamma(2k+\nu) = (2k+\nu)^2 - \nu^2 = 4k(k+\nu)$$

$$= \frac{-a_{2(k-1)}}{4k(k+\nu)}$$

$$a_2 = -\frac{1}{4\cdot 1(1+\nu)} a_0$$

$$a_4 = -\frac{1}{4\cdot 2(2+\nu)} a_2 = \frac{1}{4^2(2\cdot 1)(2+\nu)(1+\nu)} a_0$$

$$a_6 = \frac{-1}{4^3(3\cdot 2\cdot 1)(3+\nu)(2+\nu)(1+\nu)} a_0$$

$$a_{2k} = \frac{(-1)^k a_0}{4^k k!\,(k+\nu)(k+\nu-1)\cdots(1+\nu)} =$$

$$= \frac{(-1)^k \Gamma(\nu+1)\, a_0}{4^k k!\, \Gamma(\nu+k+1)}$$

$$y_1 = a_0 \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{4^k k!\, \Gamma(\nu+k+1)}$$

$$= a_0 \Gamma(\nu+1) 2^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

$$\left(a_0 = \left(\Gamma(\nu+1) 2^{\nu}\right)^{-1}\right)$$

DEFINE J_ν

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

$$\boxed{\text{BESSEL FUNCTION OF ORDER } \nu.}$$

2ND SOLUTION

$\nu \neq$ INTEGER OR HALF INTEGER

$r = -\nu$ NO TROUBLE

$$J_{-\nu} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1-\nu)} \left(\frac{x}{2}\right)^{2k-\nu}$$

FACT: IF x IS NOT 0 OR A NEGATIVE INTEGER:

$\Gamma(x)$ CAN BE DEFINED