SUMMARY OF THE METHOD OF FROBENIUS

Consider the linear, homogeneous, second order equation:

\[ y'' + p(x)y' + q(x)y = 0. \]  \hspace{1cm} (1)

Suppose that \( x = 0 \) a regular singular point:

\[ xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad |x| < R_1, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad |x| < R_2, \quad R_1, R_2 > 0. \]

Define \( \gamma(r) = r(r - 1) + p_0 r + q_0 \); the indicial equation is

\[ \gamma(r) = 0, \quad \text{roots } r_1, r_2. \]

**Case (i).** \( r_1 \) and \( r_2 \) are distinct and do not differ by an integer. There are two linearly independent solutions:

\[ y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \quad a_0 = b_0 = 1. \]  \hspace{1cm} (2)

**Case (ii).** \( r_1 = r_2 \). There is one solution \( y_1(x) \) of the form given in (2), and a second solution with the form

\[ y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n x^n. \]  \hspace{1cm} (3)

**Case (iii).** \( r_1 = r_2 + m, m \) a positive integer. There is one solution \( y_1(x) \) as in (2), and a second solution with the form

\[ y_2(x) = Cy_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \quad b_0 = 1. \]  \hspace{1cm} (4)

The constant \( C \) may or may not be zero. One may assume that \( b_m = 0 \); see below.

**FURTHER COMMENTS**

1. **Normalization.** In these formulas we have “normalized” the solutions by choosing \( a_0 \) and \( b_0 \) to have value 1. We could just as well have said only that they were nonzero, but it is convenient to have the solutions \( y_1(x) \) and \( y_2(x) \) completely defined.

2. **Radius of convergence.** All the power series in (2)–(4) are guaranteed to have radius of convergence at least as big as the smaller of \( R_1 \) and \( R_2 \).
3. Solution procedure, Case (i). The coefficients $a_n$ of the solution $y_1(x)$ are determined by substituting the given expression (2) for $y_1(x)$ into (1) and then solving successive equations for $a_1, a_2, \ldots$. These have the form (before we set $a_0 = 1$)

$$\gamma(n + r_1)a_n = \text{a linear combination of } a_0, a_1, \ldots, a_{n-1}. \quad (5)$$

The coefficients $b_n$ of the second solution $y_2(x)$ in Case (i) are found similarly.

4. Solution procedure, Cases (ii) and (iii). In these cases one first finds $y_1(x)$. The solution $y_2(x)$ of (3) or (4) can be written as $y_2(x) = Cy_1(x)(\ln x) + u(x)$, where $C = 1$ in Case (ii) and $C$ is to be determined in Case (iii), and in each case $u$ is given by a series. Substituting this form into (1) one finds that $u(x)$ must satisfy the equation

$$u'' + p(x)u' + q(x)u = \frac{C}{x^2} [y_1(x) - xp(x)y_1(x) - 2xy_1'(x)]. \quad (6)$$

One then substitutes the form of the series for $u(x)$, as given in (3) or (4), into (6) and solves for $b_1, b_2, \ldots$ and, in Case (iii), for $C$. The general structure of the equations will be similar to (5):

$$\gamma(n + r_2)b_n = \text{a linear combination of } C \text{ and } b_1, b_2, \ldots, b_{n-1}. \quad (7)$$

Recall that $C = 1$ in Case (ii). In Case (iii) the constant $C$ first appears on the right hand side of (7) when $m = n$; then $\gamma(m + r_2) = \gamma(r_1) = 0$ so that the left hand side vanishes (and $b_m$ is not determined). Then $C$ must be chosen to make the right hand side vanish also.

5. Additional free constants. Notice that there is no $b_0$ coefficient in (3). One could include a $b_0$ term in the solution, but the value of $b_0$ would not be determined by the equations; $b_0$ could be chosen freely. Choosing a nonzero value for the $b_0$, however, would amount to adding a multiple of $y_1(x)$ to the solution $y_2(x)$ as given in (3).

The situation for Case (iii) is similar. The coefficient $b_m$ in (4) will not be determined during the solution process, and it is simplest to choose $b_m = 0$. Choosing a nonzero value for $b_m$ again amounts to adding a multiple of $y_1(x)$ to the solution.

6. An ordinary point. An ordinary point of a differential equation may be considered, in some sense, as a special case of a regular singular point. If $x = 0$ is an ordinary point of (1) then the above analysis applies; one finds that $\gamma(r) = r(r-1)$ and hence that $r_1 = 1$ and $r_0 = 0$: we are in Case (iii). However, we already know that in this case there are two linearly independent solutions, as power series in $x$, which do not contain $\ln x$; this means that necessarily $C = 0$. 
