Taylor series (with radii of convergence given):

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_0^\infty x^n, \qquad |x| < 1$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_0^\infty \frac{x^n}{n!}, \qquad |x| < \infty$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_0^\infty \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

The Gamma function. For $x > 0$, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$.

If x is not 0 or a negative integer, $\Gamma(x+1) = x\Gamma(x)$.

If n is a non-negative integer, $\Gamma(n+1) = n!$. $\Gamma(1/2) = \sqrt{\pi}$.

The Method of Frobenious—solution forms: $y_1(x) = x^r \sum_{n=0}^\infty a_n x^n$,

$$y_2(x) = y_1(x)(\ln x) + x^{r_1} \sum_{n=1}^\infty b_n x^n, \qquad y_2(x) = C y_1(x)(\ln x) + x^{r_2} \sum_{n=0}^\infty b_n x^n.$$

Bessel Functions.

A. The Bessel equation of order ν: $\qquad x^2 y'' + x y' + (x^2 - \nu^2) y = 0.$

B. If u solves the Bessel equation of order ν, and $b > 0$, then

$$y(x) = x^{\nu/\alpha} u\left(\alpha\sqrt{b} x^{1/\alpha}\right) \qquad \text{solves} \qquad y'' + \frac{a}{x} y' + b x^{c-a} y = 0,$$

where

$$\alpha = \frac{2}{c-a+2} \qquad \text{and} \qquad \nu = \frac{1-a}{c-a+2}.$$

C. Bessel functions:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}$$
$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^\infty \frac{(-1)^k}{k!\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k}$$
$$Y_\nu(x) = \frac{(\cos\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}, \qquad \text{if } v \neq 0, 1, 2, \ldots.$$
$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x), \qquad \text{if } n = 0, 1, 2, \ldots.$$

The Fourier transform:

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x) e^{-i\omega x}\,dx, \qquad \mathcal{F}^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(\omega) e^{i\omega x}\,dx.$$

Various Fourier-type expansions:

We write $f(x) \sim$ series to indicate that the given series is some Fourier-type expansion of $f(x)$.

$$\left.\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \right]; \\ a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx, &\quad a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \quad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx. \end{aligned}\right\} \qquad (\mathbf{1})$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}; \qquad c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{-in\pi x/\ell}\,dx. \qquad (\mathbf{2})$$

$$\left.\begin{aligned} f(x) &\sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad (0 < x < L); \\ a_0 = \frac{1}{L}\int_0^L f(x)\,dx, &\quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx. \end{aligned}\right\} \qquad (\mathbf{3})$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad (0 < x < L); \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \qquad (\mathbf{4})$$

$$f(x) \sim \sum_{n=1,3,5,\ldots} a_n \cos\frac{n\pi x}{2L} \quad (0 < x < L); \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx. \qquad (\mathbf{5})$$

$$f(x) \sim \sum_{n=1,3,5,\ldots} b_n \sin\frac{n\pi x}{2L} \quad (0 < x < L); \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx. \qquad (\mathbf{6})$$

Some antiderivatives:

$$\int e^{-ax}\sin(bx)\,dx = -\frac{e^{-ax}}{a^2+b^2}\left(a\sin(bx) + b\cos(bx)\right) + C$$

$$\int e^{-ax}\cos(bx)\,dx = \frac{e^{-ax}}{a^2+b^2}\left(-a\cos(bx) + b\sin(bx)\right) + C$$

$$\int x\cos(bx)\,dx = \frac{\cos(bx)}{b^2} + \frac{x\sin(bx)}{b} + C \qquad \int x\sin(bx)\,dx = \frac{\sin(bx)}{b^2} - \frac{x\cos(bx)}{b} + C$$

Some trig identities:

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos(A)\cos(B) = (1/2)\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin(A)\cos(B) = (1/2)\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin(A)\sin(B) = (1/2)\left[\cos(A-B) - \cos(A+B)\right]$$

Sturm-Liouville problem:

$$[p(x)y']' + q(x)y + \lambda w(x)y = 0, \qquad \langle f(x), y(x)\rangle_w = \int_0^L f(x)g(x)w(x)\,dx$$

d'Alembert's solution:

$$u(x,t) = F(x-ct) + G(x+ct) = \frac{1}{2}(f(x-ct) + f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$