Methods of Applied Mathematics— Math 527:01 Exam 1

1. Find the power series expansion (Taylor series) centered at $x_0 = 1$ for $\dfrac{1}{3-x}$ and give its radius of convergence.

2. (a) Find the Laplace transform of $f(t) = \begin{cases} 0, & \text{if } 0 \le t < 1; \\ e^{2t}, & \text{if } t \ge 1. \end{cases}$

   (b) Find the inverse transform of $\dfrac{1}{(s-3)(s^2+1)}$ in the form of a single integral.

   (c) Use partial fractions, without computing the numerical values of the constants, to express the inverse transform of $\dfrac{1}{(s+2)^2(s^2+1)}$ as a sum of functions. (The solution should have the form $c_1 u_1(t) + \cdots + c_k u_k(t)$, where the $c_1, \ldots, c_k$ are left undetermined.)

3. Use Laplace transforms to solve the initial value problem:

$$y''(t) + 2y'(t) + 3y(t) = \delta(t-2). \qquad y(0) = 1, \;\; y'(0) = -1.$$

4. Consider the equation $\quad x^2 y'' + 2x\cos(x) y' + 5y = 0.$

   (a) $x_0 = 0$ is a regular singular point. Explain why.

   (b) What is the radius of convergence of the Frobenius series solution to the equation? Explain.

   (c) Find the indicial equation. (No further analysis required.)

(continued)

5. This problem is a study of the equation

$$x^2y'' + xy' + (x - \lambda^2)y = 0, \tag{1}$$

for various values of λ. You may take as given that $x = 0$ is a regular singular point. If one substitutes $y(x) = \sum_0^\infty a_n x_{n+r}$ in equation (1) and takes $a_{-1} = 0$, one finds

$$\sum_{n=0}^{\infty} \left\{ \left((n+r)^2 - \lambda^2\right) a_n + a_{n-1} \right\} x^{n+r} = 0 \tag{2}$$

You may take this as given. Note that parts b), c), and d) of this problem may be done independently of one another.

(a) Obtain the indicial equation for r and the recursion formula for the a_n's. (Your answers will have λ in them.)

(b) (i) In this part, $\lambda^2 = 4/9$. The indicial equation has two roots $r_1 > r_2$. Find the roots and then find the first three non-vanishing terms of the Frobenius series solution corresponding to the larger root.

(ii) Again for $\lambda^2 = 4/9$, show that $y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - (1/3))} x^{n-2/3}$ is a solution corresponding to the smaller root by verifying that the root is correct and the coefficients satisfy the appropriate recursion formula.

(c) Consider now the case $\lambda = 0$. What are the forms of the two independent solutions to (1) guaranteed by Frobenius theory? (Give the general forms only. Do not compute *anything.*)

(d) In this part consider the case $\lambda^2 = 1/4$ for which the roots of the indicial equation are $1/2$ and $-1/2$. Let $y_1(x)$ be the solution corresponding to the larger root. Will there be a second solution of the form $x^{-1/2} \sum_0^\infty a_n x^n$? If there is, find its first three non-vanishing terms. If not, state the form the second solution must take.

6. When $\lambda = 0$, the equation of problem 5 is $x^2y'' + xy' + xy = 0$. Find the general solution to this equation in terms of Bessel functions. Find constants in this general solution to get a solution with the propery that $\lim_{x \downarrow 0} y(x) = 1$.