EXERCISES II

1. Exercises from Wheeden & Zygmund, pp. 31–32: 1, 6, 10, 15, 18.

2. Mean-value Theorem #1: For \([a, b] \subseteq \mathbb{R}\), suppose \(g : [a, b] \rightarrow \mathbb{R}\) is an increasing function and \(f : [a, b] \rightarrow \mathbb{R}\) is integrable (in either sense) with respect to \(dg\). Let \(m = \inf f([a, b])\) and \(M = \sup f([a, b])\).

   (a) Show that there is some \(c \in [m, M]\) for which
   \[
   \int_a^b f(t) \, dg(t) = c \cdot [g(b) - g(a)] .
   \]

   (b) Show that if \(f \in \mathcal{C}[a, b]\), then there exists some \(\xi \in [a, b]\) for which
   \[
   \int_a^b f(t) \, dg(t) = f(\xi) \cdot [g(b) - g(a)] .
   \]

3. Mean-value theorem #2: Replacing the hypotheses of 1 above by the assumption that \(f : [a, b] \rightarrow \mathbb{R}\) is a monotonic function and \(g : [a, b] \rightarrow \mathbb{R}\) is continuous,

   (a) Show that there exists \(\xi \in [a, b]\) for which
   \[
   \int_a^b f(t) \, dg(t) = f(a) \cdot [g(\xi) - g(a)] + f(b) \cdot [g(b) - g(\xi)] .
   \]

   (b) Assuming further that \(f(b) \cdot [f(a) - f(b)] > 0\), show that there exists \(\xi \in [a, b]\) for which
   \[
   \int_a^b f(t)g(t) \, dt = f(a) \cdot \int_a^{\xi} g(t) \, dt .
   \]
   \(\text{(Riemann integrals)}\)

   [Use Wheeden & Zygmund’s Prob. 15, p. 32 as a lemma.]

   (c) Show (using (b)) that if \(0 < a < b\), then
   \[
   \left| \int_a^b \frac{\sin t}{t} \, dt \right| < \frac{2}{a} .
   \]

4. Prove this minimum-hypotheses Fundamental Theorem of the Integral Calculus: If \(x \mapsto F(x)\) is continuous on \([a, b]\) and differentiable on \((a, b)\), and if \(f : [a, b] \rightarrow \mathbb{R}\) is Riemann-integrable on \([a, b]\) and \(f(x) = F'(x)\) for \(x \in (a, b)\), then
   \[
   \int_a^b f(x) \, dx = F(b) - F(a) .
   \]