Solutions for 640:501, Real Analysis

42. If $\mu_1, \ldots, \mu_n$ are measures and $a_0, \ldots, a_n$ are non-negative constants, it is easy to check that $\sum_{i=1}^{n} a_i \mu_i$ is a measure, because countable additivity is easy to check for the sum.

If $\{E_i\}$ are sets of $\mathcal{M}$ of the measurable space $(X, \mathcal{M}, \mu)$, then for each $n$, the monotonicity of $\mu$ implies that $\mu(\bigcap_{i=n}^{\infty} E_i) \leq \mu(E_j)$ for every $j \geq n$. Hence, $\mu(\bigcap_{i=n}^{\infty} E_i) \leq \mu(E_j) \leq \inf_{i \geq n} \mu(E_i)$.

The sequence of sets $\bigcap_{i=n}^{\infty} E_i$, $n \geq 1$ increases up to $\liminf E_j$. Hence by continuity from below of measures,

$$\mu(\liminf E_j) = \lim_{n \to \infty} \mu(\bigcap_{i=n}^{\infty} E_i) \leq \lim_{n \to \infty} \inf_{i \geq n} \mu(E_i) = \liminf \mu(E_j).$$

A similar proof, but using continuity from above, works to show the corresponding fact concerning limsup, under the added assumption that $\mu(\bigcup E_i) < \infty$.

Finally, we prove $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$. This follows from finite additivity of $\mu$;

$$\mu(E) + \mu(F) = \mu(E \cap F) + \mu(E \cap F^c) + \mu(F) = \mu(E \cap F) + \mu(E \cup F).$$

42. We know that a measure is continuous from below. Conversely, suppose $\mu$ is a finitely additive measure on the $\sigma$-algebra $\mathcal{M}$ and also that $\mu$ is continuous from below. Let $\{A_n\} \subset \mathcal{M}$ be a disjoint sequence of measurable sets. Let $B_n = \bigcup_{i=1}^{n} A_i$. By finite additivity of $\mu$, we have $\mu(B_n) = \sum_{i=1}^{n} \mu(A_i)$. Now $B_n$ is an increasing sequence of measurable sets whose union is $\bigcup_{i=1}^{\infty} A_i$. So, using continuity from below in the last equality of the expression that follows,

$$\sum_{i=1}^{\infty} \mu(A_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(A_i) = \lim_{n \to \infty} \mu(B_n) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

Therefore $\mu$ is countably additive and hence is a measure.

Suppose now that $\mu(X) < \infty$ and $\mu$ is continuous from above and finitely additive. Given a disjoint sequence $\{A_n\} \subset \mathcal{M}$ of measurable sets, we want to show that $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. It suffice to show that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \mu(X) - \sum_{i=1}^{\infty} \mu(A_i).$$

Now $U_n = (\bigcup_{i=1}^{n} A_i)^c$ is a decreasing sequence of sets whose intersection is $(\bigcup_{i=1}^{\infty} A_i)^c$. Also by finite additivity, $\mu(U_n) = \mu(X) - \sum_{i=1}^{n} \mu(A_i)$. By continuity from above, which applies without problem because $\mu(X) < \infty$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \lim_{n \to \infty} \mu(U_n) = \lim_{n \to \infty} \mu(X) - \sum_{i=1}^{n} \mu(A_i) = \mu(X) - \sum_{i=1}^{\infty} \mu(A_i).$$

44. (a) Let $E \subset X$. If $\mu^*(E) = \infty$ there is nothing to prove, so assume $\mu^*(E) < \infty$. By definition of outer measure, if $\epsilon > 0$, there exists a sequence $\{A_i\}$ of sets in the algebra $\mathcal{A}$ such that $E \subset \bigcup_{i=1}^{\infty} A_i$ and $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \epsilon$. Let $A = \bigcup_{i=1}^{\infty} A_i$. Then $A \in \mathcal{A}$.
By monotonicity and subadditivity of an outer measure, \( \mu^*(E) \leq \mu(A) \leq \sum_1^\infty \mu(A_i) + \epsilon \).

We did not use in this part the assumption that \( \mu_0 \) is a premeasure.

(b) Suppose \( \mu^*(E) < \infty \). For each \( n \), let \( B_n \in \mathcal{A}_\sigma \) such that \( E \subset B_n \) and \( \mu^*(B_n) < \mu^*(E) + 1/n \). Let \( B = \cap_1^\infty B_n \); it belongs to \( \mathcal{A}_{\sigma \delta} \). Then by the monotonicity of \( \mu^* \), \( \mu^*(E) \leq \mu^*(B) \leq \mu^*(E) \), so \( \mu^*(B) = \mu^*(E) \). Suppose \( E^* \) is \( \mu^* \)-measurable. Because \( \mu_0 \) is a premeasure, the sets in \( (\sigma A) \) are also \( \mu^* \)-measurable; in particular, \( B \) is \( \mu^* \)-measurable. Hence, \( \mu^*(E) = \mu^*(B) = \mu^*(E) + \mu^*(B - E) \), and it follows that \( \mu^*(B - E) = 0 \).

Conversely, suppose \( B \in \mathcal{A}_{\sigma \delta} \), \( E \subset B \), \( \mu^*(E) < \infty \), and \( \mu^*(B - E) = 0 \). We know that \( B \) is \( \mu^* \)-measurable (Proposition 1.13) and that \( (X, \mathcal{M}_{\mu^*}, \mu^*) \) is complete (Theorem 1.11). Hence \( E \) must be \( \mu^* \)-measurable.

(c) Suppose that \( \mu_0 \) be \( \sigma \)-finite. Then there is a disjoint, countable partition \( \{U_n\} \) of \( X \) such that \( \mu^*(U_n) = \mu_0(U_n) < \infty \) for every \( n \). Suppose that \( E \) is \( \mu^* \)-measurable. Let \( E_k = E \cap U_k \). For every pair of positive integers \( k \) and \( n \), there is a \( \tilde{B}_{n,k} \in \mathcal{A}_\sigma \) such that \( E_k \subset \tilde{B}_{n,k} \) and \( \mu^*(E_k) \leq \mu^*(\tilde{B}_{n,k}) < \mu^*(E_k) + 1/n \). Set \( B_{n,k} = \tilde{B}_{n,k} \cap U_k \). Then, again, \( B_{n,k} \in \mathcal{A}_\sigma \), \( E_k \subset B_{n,k} \) and \( \mu^*(E_k) \leq \mu^*(B_{n,k}) < \mu^*(E_k) + 1/n \). Moreover, if \( j \neq k \) all the sets in \( \{B_{n,j}; n \geq 1\} \) are disjoint from those in \( \{B_{n,j}; n \geq 1\} \). Set \( B = \bigcap_n \bigcup_k B_{n,k} \); this is in \( \mathcal{A}_{\sigma \delta} \). By the disjointness of the sets for different \( k \) and \( j \), \( B - E = \bigcup_k \left( \bigcap_n B_{n,k} \right) - E \).

Each set in this union has zero measure and so \( B - E \) has zero measure.

45. If \( E \) is \( \mu^* \)-measurable and \( \mu^*(X) = \mu_0(X) < \infty \), then, by additivity of \( \mu^* \) on \( \mu^* \)-measurable sets, \( \mu^*(E) = \mu^*(X) - \mu^*(E^c) = \mu_0(X) - \mu^*(E^c) = \mu_0(X) - \mu^*(E^c) \).

Conversely, suppose \( \mu^*(E) = \mu_0(X) - \mu^*(E^c) \). By following the construction in parts (a) and (b) of the previous exercise, we know there exists a \( B \in \mathcal{A}_{\sigma \delta} \) such that \( E \subset B \) and \( \mu^*(B) = \mu^*(E) \). Likewise, there exists a \( C \in \mathcal{A}_{\sigma \delta} \) such that \( E^c \subset C \) and \( \mu^*(C) = \mu^*(E^c) = \mu(X) - \mu^*(E) \); since \( C \) is \( \mu^* \)-measurable, \( \mu^*(C) = \mu^*(X) - \mu^*(C) = \mu^*(E) = \mu^*(B) \). But since \( B \) and \( C \) are \( \mu^* \)-measurable, \( \mu(B - C^c) = \mu(B) - \mu^*(C^c) = 0 \). It follows also that \( \mu^*(B - E) = 0 \). But then we know from the previous problem that \( E \) is \( \mu^* \)-measurable.

46. Let \( \mu \) be a \( \sigma \)-finite measure on the \( \sigma \)-algebra \( \mathcal{M} \). Let \( \mu^* \) be the outer measure induced by \( \mu \), \( \mathcal{M}^* \) the \( \mu^* \)-measurable sets and \( \hat{\mu} \) the measure \( \mu^* \) restricted to \( \mathcal{M}^* \). We want to show \( \hat{\mu} \) on \( \mathcal{M}^* \) is the completion of the measure \( \mu \) defined on \( \mathcal{M} \). The following preliminary result is helpful.

**Lemma 1** Let \( X, \nu \) be a measure space, and let \( \bar{\mathcal{F}} \) be the completion of \( \mathcal{F} \) according to the definition

\[
\bar{\mathcal{F}} := \{ E \cup F; E \in \mathcal{F}, F \subset N \text{ for some } N \in \mathcal{F} \text{ such that } \nu(N) = 0 \}
\]

Then,

\[
\bar{\mathcal{F}} := \{ G - D; G \in \mathcal{F}, D \subset N \text{ for some } N \in \mathcal{F} \text{ such that } \nu(N) = 0 \}
\]

To prove this lemma consider \( E \cup F \in \bar{\mathcal{F}} \), where \( E \in \mathcal{F} \) and \( F \subset N \) for some \( N \in \mathcal{F} \) such that \( \nu(N) = 0 \). Then \( E \cup F = E \cup N - (N - (F \cup E)) \). Note that \( E \cup N \in \mathcal{F} \) and \( N - (F \cup E) \subset N \).
Conversely, consider $G - D$, where $G \in \mathcal{F}$ and there is $N \in \mathcal{F}$ with $\mu(N) = 0$ such that $D \subset N$. Then $G - D = (G - N) \cup ((N \cap G) - D)$, and $G - N \in \mathcal{F}$ and $(N \cap G) - D \subset N$. This completes the proof of the lemma.

Recall that $\overline{\mathcal{M}} := \{G \cup F; B \in \mathcal{M} \subset N \text{ for some } N \in \mathcal{M} \text{ such that } \mu(N) = 0\}$ and for such a set $G \cup F$ as in this definition, $\overline{\mu}(G \cup F) := \mu(G)$.

We wish to show that $\mathcal{M}^* = \overline{\mathcal{M}}$ and $\mu^*(E) = \overline{\mu}(E)$ for $E \in \mathcal{M}^*$. We know from problem 44 that if $E$ is $\mu^*$-measurable, there exists $B$ in $\mathcal{M}_{\sigma\delta}$ such that $E \subset B$ and $\mu^*(B - E) = 0$. But since $\mathcal{M}$ is a $\sigma$-algebra, $\mathcal{M}_{\sigma\delta} = \mathcal{M}$ and $B \in \mathcal{M}$. By the same token, since $B - E$ is $\mu^*$-measurable, there exists $C \in \mathcal{M}$ such that $B - E \subset C$ and $\mu^*(C) = \mu(C) = 0$. Since $E = B - (B - E)$, where $B \in \mathcal{M}$, it follows from the lemma that $E \in \overline{\mathcal{M}}$. This proves that $\mathcal{M}^* \subset \overline{\mathcal{M}}$.

The converse, $\overline{\mathcal{M}} \subset \mathcal{M}^*$, is true because any set of outer measure zero is in $\mathcal{M}^*$. Thus if $G \in \mathcal{M} \subset \mathcal{M}^*$ and $F \subset N$, where $N \in \mathcal{M}$ and $\mu(N) = 0$, $F \in \mathcal{M}^*$ also, and hence $G \cup F \in \mathcal{M}^*$.

Finally, if $G \in \mathcal{M}$ and $F \subset N$, where $N \in \mathcal{M}$ and $\mu(N) = 0$, then we have that $\mu(G) = \mu^*(G) \leq \mu^*(G \cup F) \leq \mu^*(G) + \mu^*(F) = \mu^*(G) = \mu(G)$. Thus $\mu^*(G \cup F) = \mu(G) = \overline{\mu}(G)$.

47. By subadditivity, $\mu(\cup_1^\infty A_i) \leq \sum_1^\infty \mu(A_i) < \infty$. Therefore we can apply continuity from above: since the sequence of sets, $B_n = \cup_1^n A_i$, $n \geq 1$, is decreasing and $\mu(B_1) < \infty$,

$$\mu(\limsup A_n) = \lim_{n \to \infty} \mu(\bigcup_n A_i).$$

However $\mu(\bigcup_n A_i) \leq \sum_n \mu(A_i)$, which converges to 0 as $n \to \infty$ because $\sum_1^\infty \mu(A_i) < \infty$.

48. Note that $\overset{\circ}{\mu}_G((0,1]) = 2$. However, for every $0 < \epsilon < 1$, $\overset{\circ}{\mu}_G((\epsilon,1]) = 1 - \epsilon$. So if $\epsilon_n \downarrow 0$, where each $\epsilon_n > 0$, $\lim_{n \to \infty} \mu((\epsilon_n,1]) = 1 \neq \mu((0,1])$. 

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