Solutions for 640:501, Real Analysis

31. Let $u$ be an increasing function on $\mathbb{R}$. For every real $a$, $\{x; u(x) > a\}$ is either an interval of the type $(b, \infty)$ or $[b, \infty)$. Either way $\{x; u(x) > a\}$ is Borel measurable. It follows by Proposition 2.3 in Folland that $u$ is measurable. If $f$ is of bounded variation on any finite interval, $f$ can be written as the difference of increasing functions. As sums of Borel measurable functions are Borel measurable, it follows that $f$ is also Borel measurable.

33. The problem statement has a typo; $A_\infty$ should be $A_0$. Also, it must be stipulated that $A_0$ contains the empty set. By Proposition 1.7 in Folland it suffices to show that $A_1$ is an elementary family, that is, it contains the empty set (which we stipulate as an assumption), it is closed under finite intersections, and that if $B \in A_1$, $B^c$ is a finite disjoint union of sets of $A_1$. It is obvious by the definition of $A_1$ that it is closed under finite intersections. By definition, every set in $A_1$ has the form, $\bigcap_1^n B_i$, where $n$ is a positive integer and $B_1, \ldots, B_n \in \{A; A \in A_0 \text{ or } A^c \in A_0\}$. We must show

$$\left[\bigcap_1^n B_i\right]^c$$

is a finite disjoint union of sets in $A_1$.

This is shown by induction on $n$. It is obviously true for $n = 1$. Suppose it is true for $n - 1$. Then

$$\left[\bigcap_1^n B_i\right]^c = \left[\bigcap_1^{n-1} B_i\right]^c \cup B_n^c = \left[\bigcap_1^{n-1} B_i\right]^c \cap B_n \cup B_n^c.$$

By the induction hypothesis, there are disjoint sets $U_1, \ldots, U_k$ in $A_1$ whose union is $\left[\bigcap_1^{n-1} B_i\right]^c$. Each set $B_n \cap U_i$ is also in $A_1$. Thus

$$\left[\bigcap_1^n B_i\right]^c = \left[\bigcup_1^k U_i \cap B_n\right] \cup B_n^c.$$

This is a disjoint union of sets in $A_1$, and hence the induction step is completed.

34. (a) Let $A_1, A_2, \ldots$ be a sequence of pairwise non-identical sets in $\mathcal{F}$. For any subset $I$ of the positive integers, define

$$B_I = \left[\bigcap_{i \in I} A_i\right] \cap \left[\bigcap_{j \in I^c} A_j^c\right].$$

Each $B_I$ is in $\mathcal{F}$. If $J$ is another subset of the positive integers and $I \neq J$ then $B_I \cap B_J = \emptyset$. Observe also that for each $i$, $A_i = \bigcap_{I : i \in I} B_I$. Since there are an infinite number of different sets $A_i$, there must be an infinite number of non-empty sets in
the collection \( \{B_I : I \subseteq \mathbb{N}\} \). (If the latter collection were finite, the set of all unions of of its elements, which includes \( \{A_i : i \geq 1\} \), would be finite, contrary to assumption.)

(b) Let \( A_1, A_2, \ldots \) be a countable infinite sequence of disjoint sets in \( \mathcal{F} \). For each different subset \( I \) of the positive integers, \( \bigcup_{i \in I} A_i \) is a different subset of \( \mathcal{F} \). The cardinality of the number of subsets of the positive integers is the same as the cardinality of the real numbers.

**37. (a)** We want to show that if \( f, g : X \to \mathbb{R} \) is measurable, then so is \( fg \). There are several proofs. For example, for a proof ‘from scratch,’ it suffices to show that \( \{x : f(x)g(x) > a\} \) is measurable for every real \( a \). If \( a > 0 \), then, letting \( Q_+ \) denote the positive rational numbers,

\[
\{f(x)g(x) > a\} \cap \{g(x) > 0\} = \{f(x) > a/g(x)\} \cap \{g(x) > 0\} = \bigcup_{r \in Q_+} \{f(x) > r > a/g(x) > 0\}
\]

Then, using a similar decomposition of \( \{f(x)g(x) > a\} \cap \{g(x) < 0\} \), where \( Q_- \) denotes the negative rational numbers,

\[
\{x : f(x)g(x) > a\} = \left[ \bigcup_{r \in Q_+} \{f(x) > r\} \cap \{g(x) > a/r\} \right] \cup \left[ \bigcup_{r \in Q_-} \{f(x) < r\} \cap \{g(x) < a/r\} \right]
\]

Since \( f \) and \( g \) are measurable, all the sets on the right-hand side are measurable, and since the right hand side is composed of a countable union of these sets, it follows that \( \{x : f(x)g(x) > a\} \) is measurable. Since \( \{f(x)g(x) > 0\} = \{f(x) > 0\} \cap \{g(x) > 0\} \cup \{f(x) < 0\} \cap \{g(x) < 0\} \), this set is also measurable, and \( \{f(x)g(x) = 0\} = \{f(x) = 0\} \cup \{g(x) = 0\} \) is measurable as well. Therefore it remains only to show that \( \{a < f(x)g(x) < 0\} \) is measurable for \( a < 0 \), but this is done in a very similar manner that the reader can supply.

An easier method is to write \( fg = h_1 + h_2 + h_3 \) where \( h_1 = (f\chi_{|f|<\infty})(g\chi_{|g|<\infty}) \), \( h_2 \) is the function which equals \( \infty \) on

\[
[\{f > 0\} \cap \{g = \infty\}] \cup [\{f < 0\} \cap \{g = -\infty\}] \cap [\{g > 0\} \cap \{f = \infty\}] \cup [\{g < 0\} \cap \{f = -\infty\}]
\]

And \( h_3 \) is defined similarly to be \(-\infty\) where \( fg = -\infty \) and otherwise \( 0 \). The function \( h_1 \) is measurable by Proposition 2.6. It is easy to show that \( h_2 \) and \( h_3 \) are also measurable and that the sum of all three is measurable too.

**38.** We suppose that \( f : X \to \mathbb{R} \) has the property that \( f^{-1}((r, \infty)) \in \mathcal{M} \) for all rational \( r \). For any real \( a \),

\[
f^{-1}((a, \infty)) = \bigcup_{r \in Q, r > a} f^{-1}((r, \infty))
\]

as the right side is a countable union of sets in \( \mathcal{M} \), \( f^{-1}((a, \infty)) \) is in \( \mathcal{M} \) also. It follows that \( f \) is measurable.
Now suppose that for each $\alpha \in \mathbb{R}$ there is a set $E_{\alpha} \in \mathcal{M}$ and $E_{\alpha} \subset E_{\beta}$ whenever $\alpha < \beta$, and $\bigcup_{\alpha \in \mathbb{R}}E_{\alpha} = X$, $\bigcap_{\alpha \in \mathbb{R}}E_{\alpha} = \emptyset$. For each $x$, define $f(x) = \sup\{\alpha; x \notin E_{\alpha}\}$. The hypothesis guarantees that the set $\sup\{\alpha; x \in E_{\alpha}^c\}$ is non-empty and bounded above for each $x$ and so $f(x) \in \mathbb{R}$ for all $x$. By definition, if $x \in E_{\beta}^c$, $f(x) \geq \beta$. Clearly also, if $x \in E_{\beta}$, $\beta \geq \sup\{\alpha; x \in E_{\alpha}^c\}$ and hence $f(x) \leq \beta$. It is not hard to see that

$$\{f(x) > a\} = \bigcup_{r \in \mathbb{Q}, r > a} E_{r}^c$$

Being a countable union of sets in $\mathcal{M}$, this set is in $\mathcal{M}$ for any $a \in \mathbb{R}$. Hence $f$ is measurable. (It appears we did not really need to use the first part of the problem, namely problem 4 of Folland.)

39. Let $(x, y)$ denote a point in $\mathbb{R} \times \mathbb{R}^k$. The map $(x, y) \to (x - a)$ is continuous and hence Borel measurable for any scalar $a$. The function $f(b, \cdot) : \mathbb{R}^k \to \mathbb{R}$ is Borel measurable by assumption for each $b$. Since projections are Borel measurable, the map $(x, y) \to f(b, y)$ is Borel measurable as a map on $\mathbb{R} \times \mathbb{R}^k$. Since products and linear combinations of Borel-measurable functions are Borel measurable and since $(x, y) \to \chi[a_i, a_{i+1}](x)$ is Borel measurable, so is

$$f_N(x, y) = \lim_{N \to \infty} \sum_{i = -N}^{N} \frac{f(a_{i+1}, y)(x - a_i) - f(a_{i}, y)(x - a_{i+1})}{a_{i+1} - a_i} \chi[a_i, a_{i+1}](x)$$

Therefore

$$f_N(x, y) = \lim_{N \to \infty} \sum_{i = -N}^{N} \frac{f(a_{i+1}, y)(x - a_i) - f(a_{i}, y)(x - a_{i+1})}{a_{i+1} - a_i} \chi[a_i, a_{i+1}](x)$$

is Borel measurable.

40. Let $\Omega := \{0, 1\}^\infty$ be the space of all sequences $\omega = (\omega_1, \omega_2, \ldots)$ of 0s and 1s. Let $\mathcal{F}$ be the $\sigma$-algebra generated by the cylinder sets of $\Omega$; a cylinder set is a subset of the form $\{\omega; (\omega_1, \ldots, \omega_n) \in B\}$, where $n$ is a positive integer and $B$ is a subset of $\{0, 1\}^n$. We want to show that the set

$$\left\{\omega; \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = \frac{1}{2}\right\}$$

is in $\mathcal{F}$.

Observe that

$$\left\{\omega; \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = \frac{1}{2}\right\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{\omega; \left| \sum_{i=1}^{n} \omega_i - \frac{1}{2}\right| < \frac{1}{k}\right\}.$$
41. Let $\mathcal{C}$ be an algebra and let $\mathcal{M}$ be a monotone class containing $\mathcal{C}$. We want to show that $\sigma(\mathcal{C}) \subset \mathcal{M}$. For this, we may assume that $\mathcal{M}$ is the smallest monotone class containing $\mathcal{C}$.

Let $\mathcal{G} := \{ A \in \mathcal{M}; A^c \in \mathcal{M} \}$. This is a monotone class. If $A_1, A_2, \ldots$ are in $\mathcal{G}$ with $A_1 \subset A_2 \subset \cdots$, then $A_1^c \supset A_2^c \supset \cdots$, $A_i^c \in \mathcal{M}$ for every $i$ and hence, since $\mathcal{M}$ is a monotone class, $[\cup A_n]^c = \cap A_n^c$ is also in $\mathcal{M}$. Therefore $\mathcal{G}$ is closed under increasing limits. A similar argument shows that $\mathcal{G}$ is closed under decreasing limits, and hence that $\mathcal{G}$ is a monotone class.

Since $\mathcal{G}$ contains $\mathcal{C}$, by assumption, it follows by the assumed minimality of $\mathcal{M}$ that $\mathcal{M} \subset \mathcal{G}$. This proves $\mathcal{M}$ is closed under complements.

Fix any $A \in \mathcal{C}$ and let $\mathcal{F}_A = \{ B \in \mathcal{M}; A \cup B \in \mathcal{M} \}$. It is easy to show that $\mathcal{F}_A$ is a monotone class containing $\mathcal{C}$. Thus $\mathcal{F}_A = \mathcal{M}$, implying that $A \cup B \in \mathcal{M}$ whenever $A \in \mathcal{C}$ and $B \in \mathcal{M}$.

Now fix any $B \in \mathcal{M}$. Again define $\mathcal{F}_B = \{ A \in \mathcal{M}; A \cup B \in \mathcal{M} \}$. The result of the previous paragraph implies that $\mathcal{C} \subset \mathcal{F}_B$. But $\mathcal{F}_A = \{ B \in \mathcal{M}; A \cup B \in \mathcal{M} \}$ is again a monotone class and so $\mathcal{M} \subset \mathcal{F}_B$. As this is true for any $B \in \mathcal{M}$, it follows that $\mathcal{M}$ is closed under finite unions.

We have therefore proved that the monotone class $\mathcal{M}$ is an algebra. But a monotone class that is an algebra must be a $\sigma$-algebra. Hence $\sigma(\mathcal{C}) \subset \mathcal{M}$. (Observe that $\mathcal{M} \subset \sigma(\mathcal{C})$ because any $\sigma$-algebra is a monotone class and we are assuming $\mathcal{M}$ is the minimal monotone class containing $\mathcal{C}$. Thus we have actually proved $\sigma(\mathcal{C}) = \mathcal{M}$.)