76. The integrand $(1 - xy)^{-a}$ is positive on $[0, 1]^2$. Therefore Tonelli’s theorem says that the iterated integrals are equal and are equal to the integral of $(1 - xy)^{-a}$ over $[0, 1]^2$ with respect to 2-dimensional Lebesgue measure. (This is enough for the solution. Many of you went on to analyze convergence of the integral, but don’t confuse finiteness of an integral with existence of an integral; an integral can exist and be infinite.)

(Analysis of convergence (one method): Observe that if $(x, y) \in (0, 1)^2$,

\[(1 - xy)^{-a} = \sum_{n=0}^{\infty} \frac{a(a + 1) \cdots (a + n - 1)}{n!} (xy)^n = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a) \Gamma(n + 1)} (xy)^n.\]

By the dominated convergence theorem

\[\int_{[0,1]^2} \frac{1}{(1 - xy)^a} dm^2 = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a) \Gamma(n + 1)} \frac{1}{(n + 1)^2}.\]

By Stirling’s formula, $\Gamma(z) \sim (\sqrt{2\pi/z}) (z/e)^z$, as $z \to \infty$, and from this

\[\frac{\Gamma(a + n)}{\Gamma(a) \Gamma(n + 1)} \sim \frac{(a + n)^{a-1}}{\Gamma(a)}, \quad z \to \infty.\]

Thus, \(\int_{[0,1]^2} \frac{1}{(1 - xy)^a} dm^2\) converges if and only if

\[\sum_{n=0}^{\infty} \frac{(a + n)^{a-1}}{(n + 1)^2} < \infty,\]

if and only if $0 < a < 2$.)

(c) For every $0 < y \leq 1$, the integral $\int_0^1 \chi_{\{y < |x - .5|\}} (x - .5)^{-3} dx$ is defined and equal to 0, as the integrand is odd about $x = .5$. Thus

\[\int_0^1 \left[ \int_0^1 \chi_{\{y < |x - .5|\}} (x - .5)^{-3} dx \right] dy = 0.\]

On the other hand,

\[\int_0^1 \chi_{\{y < |x - .5|\}} |x - .5|^{-3} dy = \text{sgn}(x - .5)(x - .5)^{-2}\]

where $\text{sgn}(z) = 1$ if $z \geq 0$ and $\text{sgn}z = -1$ if $z < 0$. As a function of $x$ this is not integrable over $[0, 1]$ so the iterated integral

\[\int_0^1 \left[ \int_0^1 \chi_{\{y < |x - .5|\}} (x - .5)^{-3} dy \right] dx\]
is not defined. This does not contradict Fubini’s theorem, because by Tonelli’s theorem. Indeed,
\[
\int_{[0,1]^2} f^+ \, dm^2 = \int_{[0,1]^2} f^- \, dm^2 = \infty
\]
Hence the integral \( \int_{[0,1]^2} f \, dm^2 \) is not defined and \( f \not\in L^1([0,1]^2) \).

79. (a)

\[
\int_0^\infty \frac{\left| \sin x \right|}{x} \, dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\left| \sin x \right|}{x} \, dx
\]
\[
> \sum_{n=0}^{\infty} \frac{1}{(n+1)\pi} \int_0^\pi \sin x \, dx = \infty.
\]

(b) If \( b \) is finite, \( \int_0^b \int_0^\infty e^{-xy}\left| \sin x \right| \, dy \, dx = \int_0^b \frac{\left| \sin x \right|}{x} \, dx < \infty \). Thus Tonelli’s theorem implies \( e^{-xy} \sin x \in L^1([0,b] \times [0,\infty)) \), and so Fubini’s theorem implies,

\[
\int_0^b \frac{\sin x}{x} \, dx = \int_0^b \int_0^\infty e^{-xy} \sin x \, dy \, dx = \int_0^\infty \int_0^b e^{-xy} \sin x \, dx \, dy
\]
\[
= \int_0^\infty \left[ \frac{1}{1+y^2} - \frac{e^{-by}}{1+y^2}(y \sin b + \cos b) \right] \, dy.
\]

By the dominated convergence theorem, as \( b \to \infty \) this converges to \( \int_0^\infty (1+y^2)^{-1} \, dy = \pi/2 \).