Problems for 640:501, Real Analysis

Hand in 61, 62, and 64.

57. Folland, Chapter 2, 13.

58. Folland, Chapter 2, 14.

59. Folland, Chapter 2, 16.

60. Folland, Chapter 2, 19.


63. Folland, Chapter 2, 27.

64. Folland, Chapter 2, 28.

65. Another approach to establishing existence of Lebesgue measure on $\mathbb{R}^d$ is through the following result.

**Theorem 1** Let $\rho$ be an outer measure on a metric space $S$. Then the Borel sets of $S$ are $\rho$-measurable if and only if $\rho(A \cup B) = \rho(A) + \rho(B)$ whenever $d(A, B) > 0$. Here $d$ is the metric and $d(A, B) = \inf\{d(x, y); x \in A, y \in B\}$.

The purpose of this problem is prove this theorem.

(a) Prove the only if part. Let $\bar{A}$ denote the closure of $A$. Assuming that Borel sets are $\rho$-measurable implies $\bar{A}$ is $\rho$-measurable. Observe that if $d(A, B) > 0$, then $d(\bar{A}, B) > 0$ also.

(b) Assume $\rho(A \cup B) = \rho(A) + \rho(B)$ whenever $d(A, B) > 0$, and prove that Borel measurable sets are $\rho$-measurable. You may use the following steps. First note that it suffices to prove that any closed set is $\rho$-measurable and hence it suffices to show that for any $E \subset S$, such that $\rho(E) < \infty$, and any closed set $A$, $\rho(A) \geq \rho(E \cap A) + \rho(E \cap A^c)$.

(Explain.)

Next let $B_1 := E \cap \{x; d(x, A) > 1\}$ and for $n \geq 2$,

$$B_n := E \cap \left\{ x; \frac{1}{n-1} \geq d(x, A) > \frac{1}{n} \right\}.$$

These are the slices of “Carathéodory’s onion.” Observe that $E \cap A^c = \bigcup_{i=1}^{\infty} B_i$ and that for every finite $n$, $d(E \cap A, \bigcup_{i=1}^{n} B_i) > 0$.

(i) Show that $\sum_{i=1}^{\infty} \rho(B_i) < \infty$. 

(Hint: The sets $B_1, B_3, \ldots$ are at a positive distance one from another and their union is contained in $E$.)

(ii) Show that $\lim_{n \to \infty} \rho(\bigcup_{j=1}^{n} B_j) = \rho(E \cap A^c)$. (Hint: Use subadditivity, $E \cap A^c = \bigcup_{i=1}^{\infty} B_i$, and (i).)

(iii) Show that for every $n$,

$$\rho(E) \geq \rho(E \cap A) + \rho\left(\bigcup_{i=1}^{n} B_i\right).$$

Use (ii) to finish the proof.

66. Let $m^*$ be Lebesgue outer measure on $\mathbb{R}$. Use the result of problem 57 to show that the Borel sets are Lebesgue measurable.