Problems for 640:501, Real Analysis

Hand in 31, 34, 40 on October 8.

31. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function which has bounded variation on every finite interval. Show that \( f \) is Borel measurable.

32. In each case decide whether the collection of sets \( \mathcal{A} \) is a \( \sigma \)-algebra, an algebra only, or neither.

   (i) \( \mathcal{A} \) is the collection of all finite and co-finite subsets of an infinite set \( S \). (A subset \( B \) of \( S \) is co-finite if \( B^c \) is finite.)

   (ii) \( \mathcal{A} \) is the collection of all countable (i.e. finite or countably infinite) and co-countable subsets of an infinite set \( S \).

   (iii) All open and closed subsets of \( \mathbb{R} \).

33. Let \( \mathcal{A}_0 \) be a non-empty collection of subsets of \( S \). Take \( \mathcal{A}_1 \) to be the set of all finite intersections of the form \( \bigcap_1^n B_i \) where for each \( i \), either \( B_i \in \mathcal{A} \) or \( B_i^c \in \mathcal{A} \). Let \( \mathcal{A}_2 \) be the collection of all finite disjoint unions of elements of \( \mathcal{A}_\infty \). Show that \( \mathcal{A}_2 \) is the smallest algebra containing \( \mathcal{A}_0 \).

34. Folland, Chapter 1, problem 3.

35. Folland, Chapter 1, problem 4.

36. Folland, Chapter 1, problem 5.

37. Folland, Chapter 2, problem 3.

38. Folland, Chapter 2, problems 4 and 7.


40. Let \( \Omega := \{0, 1\}^\infty \) be the space of all sequences \( \omega = (\omega_1, \omega_2, \ldots) \) of 0’s and 1’s. Let \( \mathcal{F} \) be the \( \sigma \)-algebra generated by the cylinder sets of \( \Omega \); a cylinder set is a subset of the form \( \{\omega; (\omega_1, \ldots, \omega_n) \in B\} \), where \( n \) is a positive integer and \( B \) is a subset of \( \{0, 1\}^n \). Show that the set

\[
\left\{ \omega; \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \omega_i = \frac{1}{2} \right\}
\]

is in \( \mathcal{F} \). (Hint: try an approach similar to the discussion in class that \( \{x; \lim f_n(x) \text{ exists}\} \in \mathcal{F} \) if \( f_n \) are all \( \mathcal{F} \)-measurable.)

41. A nonempty collection \( \mathcal{M} \) of subsets of \( S \) is called a monotone class if

   (i) If \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \) and if \( A_i \in \mathcal{M} \) for each \( i \), then \( \bigcup_1^\infty A_i \in \mathcal{M} \).

   (ii) If \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \) and if \( A_i \in \mathcal{M} \) for each \( i \), then \( \bigcap_1^\infty A_i \in \mathcal{M} \).

Prove the Monotone Class Theorem:

   Let \( \mathcal{C} \) be an algebra and suppose \( \mathcal{M} \) is a monotone class containing \( \mathcal{C} \). Then \( \sigma(\mathcal{C}) \subseteq \mathcal{M} \)

   (Hint: By problem 35, it is enough to show that \( \mathcal{M} \) is an algebra.)