19. Dini’s theorem. Let \((f_n)\) be a decreasing sequence \((f_1 \geq f_2 \geq \cdots)\) of continuous functions on a compact metric space \(X\) such that \(f(x) = \lim_{n \to \infty} f_n(x)\) exists for every \(x \in X\) and defines a continuous function on \(X\). Show that \((f_n)\) converges to \(f\) uniformly.

Hint: By considering \(g_n = f_n - f\) it suffices to prove the theorem for \(f \equiv 0\). In this case, observe that the family of open sets \(\{x; f_n(x) < \epsilon\}, n \geq 1\) is an open cover of \(X\).

20. Complete the proof of existence of solutions to an ode in in the handout on the Arzelà-Ascoli Theorem (available from the course web page) by proving equation (4).

21. Let \(\phi\) be a continuously differentiable function on \([a,b]\). (At \(a\), \(\phi'(a)\) is defined to be the right derivative, at \(b\), \(\phi'(b)\) is the left derivative.) Show: if \(\int_a^b f d\phi\) exists, then
\[
\int_a^b f d\phi = \int_a^b f \phi' dx.
\]

22. Let \(f, g, \phi, \psi\), be functions defined on the interval \([a,b]\). Assume that \(\int_a^b f d\phi\), \(\int_a^b g d\phi\), \(\int_a^b f d\psi\), and \(\int_a^b g d\psi\) are all defined in the Riemann-Stieltjes sense. Then so are then
\[
\int_a^b c_1 f + c_2 g d\phi = c_1 \int_a^b f d\phi + c_2 \int_a^b g d\phi
\]
\[
\int_a^b f d(\phi + \psi) = \int_a^b f d\phi + \int_a^b f d\psi
\]
These statements entail that the integrals on the left-hand sides are well-defined.

23. The following notation is useful: \(x \wedge y := \min\{x, y\}\), \(x \vee y := \max\{x, y\}\), \(x^+ = x \vee 0\), \((x^-) = (-x) \wedge 0\). We apply these notations to functions in the obvious way; thus \(f^+\) is the function defined by \(f^+(x) := 0 \vee f(x)\), etc. Notice that \(|x| = x^+ + x^-\).

(a) Show that if \(f\) is Riemann integrable on \([a,b]\) then so are \(f \wedge M\) and \(f \vee M\) for any finite constant \(M\).

(b) Show that if \(f\) is Riemann integrable on \([a,b]\) then so is \(|f|\).

(c) Show that if \(f\) and \(g\) are Riemann integrable on \([a,b]\) then so is \(fg\). (It is enough to prove this for \(f \geq 0\) and \(g \geq 0\). Why?)

(Statements (a) (b) (c) are all valid if Riemann integrable is replaced by Stieltjes integrable with respect to a monotone \(\phi\). Proving the Riemann case just allows you to use less notation!)