Problems for 640:501, Real Analysis

For Wednesday, September 17, do problems 9-16. Hand in 10, 12, 14. Problem 16 is optional and is for those who might like a challenge. It is not hard to guess what the limit of a Cauchy sequence in $\ell^p$ should be. It is harder to show convergence to the conjectured limit in $\ell^p$ when $p > 1$. Problems 17 and 18 will be among those for next week.

9. Let $(f_n)$ be a sequence of continuous functions on a metric space $X$, let $f$ be a function on $X$ and suppose that $\lim_{n \to \infty} \sup_X |f_n(x) - f(x)| = 0$. Show that $f$ is continuous.

10. Define the distance between two subsets $A$ and $B$ of a metric space $(X, d)$ by $d(A, B) = \inf \{d(x, y) \mid x \in A, y \in B\}$.
   (a) Give an example of two disjoint closed sets $A$ and $B$ of $\mathbb{R}^d$, with the Euclidean metric, such that $d(A, B) = 0$.
   (b) Show that if $A$ and $B$ are closed and disjoint (in any metric space) and $A$ is compact, then $d(A, B) > 0$.
   (c) Show that for any subset $A$ of $X$, $x \to d(x, A)$ is continuous in $x$.

11. Prove the following result, crucial to the proof of the contraction mapping theorem given in class. Let $(X, d)$ be a metric space and let $\phi : X \to X$ be a contraction, that is, for some $0 < \alpha < 1$, $d(\phi(x), \phi(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. Let $\phi^n$ denote $\phi$ composed with itself $n$ times. Then for any $x \in X$, the sequence $(\phi^n(x))$ is Cauchy.

12. Consider $C([0,1]; \mathbb{R})$ with the sup norm metric. Show that if $\epsilon < 1$ the unit ball $\{f : \|f\|_{\infty} \leq 1\}$ does not admit a finite covering by open balls of radius $\epsilon$. This gives another proof that the unit ball in $C([0,1]; \mathbb{R})$ is not totally bounded.

13. In class we applied the contraction mapping principle to solve the ordinary differential equation
   \[ \frac{dy}{dt} = F(y), \quad y(0) = y_0. \]
   when $F$ is a Lipschitz function. However the technique only produced a solution on $0 \leq t \leq T$ for $T < 1/K$ where $K$ is the Lipschitz constant of $F$. This restriction can be overcome with a more careful analysis. Let $\|x\|_{\infty, t} := \sup_{[0, t]} |x(s)|$.
   Assume $F$ is Lipschitz: for some constant $K$, $|F(x) - F(y)| \leq K|x - y|$ for all $x$ and $y$. Define recursively the sequence of functions:
   \[ y^{(0)}(t) \equiv y_0, \quad y^{(1)}(t) := y_0 + \int_0^t F(y^{(0)}(s)) \, ds, \ldots, y^{(n+1)} := y_0 + \int_0^t F(y^{(n)}(s)) \, ds. \]
   (This is called Picard iteration.) Show that for all positive $t$, and $n \geq 1$
   \[ \|y^{(n)} - y^{(n-1)}\|_{\infty, t} \leq \frac{K^n t^n}{n!} |y_0|. \]
   Use this to show that for any $T > 0$, the sequence $(y^{(n)})_{n \geq 1}$ is Cauchy in $C([0, T])$ with the sup norm. Show that the limit of this sequence satisfies the differential equation on $[0, T]$. 

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Problems 14 and 15 explore the concept of upper semicontinuity. Let \( \mathbb{R} \) denote the extended reals, (extended to include \(+\infty\) and \(-\infty\)). A function \( f : X \to \mathbb{R} \) on a metric space \( X \) is called upper semicontinuous if for every \( x \in X \),

\[
\limsup_{n \to \infty} f(x_n) \leq f(x) \quad \text{whenever} \quad \lim_{n \to \infty} x_n = x. \tag{1}
\]

Similarly, \( f \) is said to be lower semicontinuous if

\[
\liminf_{n \to \infty} f(x_n) \geq f(x) \quad \text{whenever} \quad \lim_{n \to \infty} x_n = x. \tag{2}
\]

14. Let \( f : X \to \mathbb{R} \) be a function on the metric space \( X \). Show that \( f \) is upper semicontinuous if and only if the set \( f^{-1}([a, \infty]) \) is closed for every real number \( a \).

15. Semicontinuity is a useful in optimization theory. Here is why. Show that if \( X \) is a compact metric space and \( f : X \to \mathbb{R} \) is upper semicontinuous, there is a point \( x^* \in X \) at which \( f \) achieves its maximum, that is

\[
f(x^*) \geq f(x) \quad \text{for all} \quad x \in X.
\]

16. \(^\star\) Show that for \( 1 \leq p < \infty \), \( \ell^p \) is complete.

17. Let \( X \) be a compact metric space and let \( C(X; R) \) denote the continuous, real-valued functions on \( X \). In class we proved this version of the Arzela-Ascoli theorem: a subset \( S \) of \( C(X; R) \) is relatively compact if (i) \( \sup \{ ||f||_\infty; f \in S \} < \infty \), and (ii) \( S \) is uniformly equicontinuous. Show that (i) and (ii) are true if the following, apparently weaker conditions are both assumed: (i') for each \( x \in X \), \( \sup \{ ||f(x)||; f \in S \} < \infty \), and (ii') for each \( x \in X \) we have that for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( d(x, y) < \delta \) implies \( |f(x) - f(y)| < \epsilon \) for all \( f \in S \).

18. Suppose that \( f \) is continuous on \([0, 1]\) and define the function \((Tf)(y)\) on \([0, 1]\) by

\[
Tf(y) = y \int_0^1 \frac{f(xy)}{5 + x} \, dx.
\]

It can be shown that \( Tf \) is continuous. Show that if \((f_n)\) is a sequence of continuous functions uniformly bounded in sup norm, then \((Tf_n)\) contains a subsequence that converges uniformly.