Let $F$ be a right-continuous, non-decreasing, real valued function on $\mathbb{R}$. For each left-half open interval $(a, b]$, define

$$\tilde{m}_F ((a, b]) \triangleq F(b) - F(a)$$

For each finite disjoint union $A = \bigcup_1^n (a_i, b_i]$, define

$$\tilde{m}_F (A) = \sum_1^n F(b_i) - F(a_i).$$

Then $\tilde{m}_F$ defines a finitely additive measure on the algebra $\mathcal{V}$ of finite disjoint unions of left-half open intervals. It is a generalization of the length measure $m$ defined on $\mathbb{R}$ in the construction of Lebesgue measure; if $F$ is the identity function $\tilde{m}_F ((a, b]) = b - a$.

The proof that $\tilde{m}_F$ is consistently defined and is finitely additive goes just like the proof for $m$ and is omitted.

**Proposition 1** $\tilde{m}_F$ is continuous from below on $(\mathbb{R}, \mathcal{V})$.

**Proof:** The proof here follows the same strategy as the proof of the continuity from below of $m$ as given in section 5.1 of the lecture notes on *Construction of Measures*, so we only put in enough details to show where the assumption that $F$ is right-continuous is used.

For the same reason as was argued for $m$, it suffices to show that $\tilde{m}_F$ is continuous from above at $\emptyset$, and this is what we shall prove. The crucial step in the proof for $m$ was to show that for each bounded $B$ in $\mathcal{V}$ and for any $\epsilon > 0$, there exists a $B'$ in $\mathcal{V}$ so that the closure $\overline{B}'$ of $B'$ is contained in $B$ and $m (B - B') < \epsilon$. The right-continuity of $F$ allows the generalization of this fact to $\tilde{m}_F$. To see this denote the representation of a bounded set $B$ in $\mathcal{V}$ as a finite disjoint union be $B = \bigcup_1^n (a_i, b_i]$. For $\delta > 0$, define $B_\delta \triangleq \bigcup_1^n (a_i + \delta, b_i]$. Observe that for all $\delta$ sufficiently small,

$$\tilde{m}_F (B - B_\delta) = \sum_1^n F(a_i + \delta) - F(a_i).$$

Therefore, given any $\epsilon > 0$, the right-continuity of $F$ implies that there is a positive $\delta$ such that $\tilde{m}_F (B - B_\delta) < \epsilon$. Clearly, $\overline{B_\delta} \subset B$.

The proof of continuity from above at $\emptyset$ can now proceed as before. Assume that $\{A_n\}$ is a sequence in $\mathcal{V}$, that $A_n \downarrow \emptyset$ and that $\tilde{m}_F (A_1) < \infty$, so that $A_1$, and hence all $A_n$ are bounded. To avoid triviality, assume also that $\tilde{m}_F (A_n) > 0$ for all $n$. 


Fix an $\epsilon > 0$, arbitrary. For each $n$, let $A'_n$ be a set in $\mathcal{V}$ such that $A'_n \subset A_n$ and $\bar{m}_F (A_n - A'_n) < \epsilon/2^n$. Let $F_n = \cap_1^n A'$. Then, as before $\bar{m}_F (A_n - F_n) < \epsilon$ and $F_n \downarrow \emptyset$. The set $F_n$ are compact, and therefore there is an $N$ so that $F_N$ is empty, and $m_F(A_n) \leq \bar{m}_F (A_N) < \epsilon$ for all $n \geq N$. Since $\epsilon$ was arbitrary, we have shown $\bar{m}_F (A_n) \downarrow 0$.

By the extension theorem, $\bar{m}_F$ admits a unique extension to a measure $m_F$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, called the Lebesgue-Stieltjes measure. Of course, this can be completed to get the measure space is $(\mathbb{R}, \mathcal{M}_F, m_F)$, where $\mathcal{M}_F$ is the $\sigma$-algebra of $m_F^*$-measurable sets, associated to the outer measure $m_F^*$ induced by $\bar{m}_F$. In general, $\mathcal{M}_F$ differs from the $\sigma$-algebra of Lebesgue measurable sets. For example if $F(x)$ is the function $\chi_{[0, \infty)}(x)$, the indicator function of $[0, \infty)$, then $\mathcal{M}_F$ is in fact the collection of all subsets of $\mathbb{R}$, and $m_F(A) = 1$ if $0 \in A$, $m_F(A) = 0$, if not.

Any measure on the Borel sets of $\mathbb{R}$ which is finite on every compact subset is actually a Lebesgue-Stieltjes measure. Indeed, let $\rho$ be a such a measure. Define

$$F(x) = \begin{cases} \rho([0, x]), & \text{if } x \geq 0; \\ -\rho((x, 0)), & \text{if } x < 0. \end{cases}$$

The continuity of $\rho$ from above and below then imply the $F$ is right-continuous, while the monotonicity of $\rho$ implies that $F$ is increasing. With this definition

$$\rho((a, b]) = F(b) - F(a).$$

Hence $\rho = m_F$.

Lebesgue-Stieltjes measures are important in probability theory. A right-continuous, increasing function such that $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$ is called a probability distribution function, or sometimes a cumulative distribution function. We think of such an $F$ as a description for an experiment that produces a random real number, in which the probability that the chosen number falls in $(a, b]$ is $F(b) - F(a)$. The requirements that $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$ are imposed so that the probability that the number falls in $(-\infty, \infty)$ is $1$, as it should be. $F$ induces the measure $m_F$ on the Borel sets of $\mathbb{R}$ which is a probability measure in the sense that $m_F(\mathbb{R}) = 1$; in probability theory, $m_F(U)$, for a Borel set $U$, is then interpreted as the probability that the randomly produced number falls in $U$. Conversely if, $P$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$,

$$F(x) \triangleq P((\infty, x])$$

defines a probability distribution function such that $P = m_F$. 