45. Let $A$ and $B$ be Lebesgue measurable subsets of $\mathbb{R}$. Show that $A \times B$ is a Lebesgue measurable subset of $\mathbb{R}^2$.

46. Let $\mathcal{R}$ be an algebra of subsets of $S$. Let $\mu$ be a $\sigma$-finite measure on $(S, \sigma(\mathcal{R}))$. Show that for each $A \in \sigma(\mathcal{R})$ and $\epsilon > 0$ there is an $A_\epsilon$ in $\mathcal{R}$ with $\mu(A \Delta A_\epsilon) < \epsilon$.

47. (From Folland) Let $\hat{m}$ be a finitely additive measure on $(S, \mathcal{R})$, where $\mathcal{R}$ is an algebra. Assume $\hat{m}(S) < \infty$ and $\hat{m}$ is continuous from below. Let $\mu^*$ be the outer measure induced by $\hat{m}$. Define the inner measure of a set by $\mu_s(A) = \mu^*(S) - \mu^*(A^c)$. Show that $A$ is $\mu^*$-measurable if and only if $\mu^*(A) = \mu_s(A)$. See problem 44 for help.

48. Prove Theorem 3 in the Lecture Notes, Construction of Measures.

49. Let
$$F(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

Let $m_F^*$ be the outer measure induced by $F$—see Lecture Notes, Lebesgue-Stieltjes Measures. Determine the class $m_F^*$ measurable sets and describe $m_F^*$.

50. Describe $m_F$ and the collection of $m_F^*$-measurable sets in the following examples.
   
   (a) $F(x) = \begin{cases} x, & \text{if } x < 0; \\ 1 + x, & \text{if } x \geq 0. \end{cases}$
   
   (b) $F(x) = \begin{cases} 0, & \text{if } x < 0; \\ x, & \text{if } 0 \leq x < 2; \\ 2, & \text{if } x \geq 2; \end{cases}$

51. (Folland) (a) Suppose $A$ is a Lebesgue measurable subset of $\mathbb{R}$ with positive Lebesgue measure. For any $\alpha$ such that $0 < \alpha < 1$, there is an open interval $I$ with $m(A \cap I) > \alpha m(I)$.

   (b) Let $A$ be a Lebesgue measurable set with positive Lebesgue measure. Then $A - A$, which is defined to be the set $\{x - y ; x \in A, y \in A\}$, contains an open interval. (Hint: If the interval $I$ is as in part (a), show that $(-m(I)/2, m(I)/2)$ is contained in $A - A$.)

52. Let $S$ be a metric space with metric $d$. An outer measure $\nu$ on $S$ is called a metric outer measure if
$$\nu(A \cup B) = \nu(A) + \nu(B) \quad \text{whenever dist}(A, B) > 0.$$ 

(Here $\text{dist}(A, B) \triangleq \inf\{d(x, y) ; x \in A, y \in B\}$.)
Show that if \( F \) is right-continuous and increasing on \( \mathbb{R} \), the outer measure \( m_F^* \) induced by \( F \) is a metric outer measure.

54. (Follow up to 53). This problem gives an alternative approach to proving that the Borel sets are \( m_F^* \)-measurable. The object is to prove:

**Theorem 1** If \( \nu \) is a metric outer measure on a metric space \( S \), then the Borel sets of \( S \) are \( \nu \)-measurable. Conversely, if the Borel sets are \( \nu \)-measurable, then \( \nu \) is a metric outer measure.

Fill in the details of the following proof.

(a) Prove that if the Borel sets are \( \nu \)-measurable, then \( \nu \) is a metric outer measure. Hint: If \( \text{dist}(A,B) > 0 \), then \( \bar{A} \cap B = \emptyset \), where \( \bar{A} \) is closed. Use the \( \nu \)-measurability of \( \bar{A} \) to break up \( \nu(A \cup B) \) into two parts.

Now assume that \( \nu \) is a metric outer measure. To prove that the Borel sets are \( \nu \)-measurable, it is enough to prove that the closed sets are Borel measurable. Thus let \( A \) be any closed set. It is necessary to show that

\[
\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c) \quad \text{for every} \ E \subseteq S.
\]

(Since the opposite inequality is automatically true by subadditivity of \( \nu \), one can then conclude equality holds.)

If \( \nu(E) = \infty \), then (1) is true trivially. Hence, assume \( \nu(E) < \infty \). Let \( G_0 = \{ x; \text{dist}(x, A) > 1 \} \cap E \). For \( n \geq 1 \), define

\[
G_n = \left\{ x; \frac{1}{n+1} < \text{dist}(x, A) \leq \frac{1}{n} \right\} \cap E.
\]

(b) Use the metric outer measure property to prove

\[
\sum_{k=0}^{\infty} \nu(G_{2k}) \leq \nu(E) < \infty, \quad \sum_{k=0}^{\infty} \nu(G_{2k+1}) \leq \nu(E) < \infty.
\]

(c) Use (b) and subadditivity to prove

\[
\lim_{n \to \infty} \nu \left( E \cap \left\{ x; \text{dist}(x, A) > \frac{1}{n} \right\} \right) = 0.
\]

Hint: \( E \cap A^c = \left\{ x; \text{dist}(x, A) > \frac{1}{n} \right\} \cup \bigcup_{m=n}^{\infty} G_m \).

(d) For any \( n \), \( E \cap A \) and \( E \cap \left\{ x; \text{dist}(x, A) > \frac{1}{n} \right\} \) are a positive distance apart and their union is contained in \( E \). Complete the proof of (1).